

POINTWISE ESTIMATES FOR EXCEEDANCE TIMES OF PERPETUITY SEQUENCES

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ABSTRACT. We consider large exceedance probabilities of the perpetuity sequence

$$Y_n = B_1 + A_1 B_2 + \cdots + (A_1 \cdots A_{n-1}) B_n,$$

where (A_n, B_n) are i.i.d. random variables with values in $\mathbb{R}^+ \times \mathbb{R}$ and the exceedance times are defined as $\tau_u = \inf\{n : Y_n > u\}$. Applying techniques based on analyzing path behavior of Y_n we provide the asymptotics of the sequence $\mathbb{P}[\tau_u = \rho \log u]$, $\rho > 0$, $u \rightarrow \infty$. This improves essentially the results of [3], where we identified probabilities $\mathbb{P}[\tau_u \in I_u]$, for some large intervals I_u around k_u , with lengths growing at least as $\log \log u$. Remarkable analogies and differences to random walks [15] are discussed.

1. INTRODUCTION

1.1. The perpetuity sequence. Let (A_n, B_n) be i.i.d. (independent identically distributed) random variables with values in $\mathbb{R}^+ \times \mathbb{R}$. We consider the perpetuity sequence

$$(1.1) \quad Y_n = B_1 + A_1 B_2 + \cdots + (A_1 \cdots A_{n-1}) B_n, \quad n = 1, 2, \dots$$

Under mild contractivity hypotheses Y_n converges a.s. to the random variable

$$(1.2) \quad Y = \sum_{n=0}^{\infty} A_1 \cdots A_n B_{n+1},$$

that is the unique solution of the random difference equation

$$(1.3) \quad Y =_d AY + B, \quad Y \text{ independent of } (A, B).$$

If we assume additionally that there is α_0 such that $\mathbb{E}A^{\alpha_0} = 1$, then

$$(1.4) \quad \lim_{u \rightarrow \infty} u^{\alpha_0} \mathbb{P}[Y > u] = c_+, \quad \lim_{u \rightarrow \infty} u^{\alpha_0} \mathbb{P}[Y < -u] = c_-,$$

for some constants c_-, c_+ satisfying $c_- + c_+ > 0$.

(1.4), proved in the fundamental works Kesten [14] and Goldie [11], has numerous consequences both for applications and purely theoretical issues. On one hand, the perpetuity sequence plays an important role in analyzing the ARCH and GARCH financial time series models, see Engle [9] and Mikosch [17]. On the other, it is connected to the weighted branching process and the branching random walk, see Guivarc'h [12], Liu [16] and Buraczewski [1].

Until recently very little has been known concerning the path properties of perpetuity sequence Y_n . Analyzing path properties, although technically quite involved, is ultimately very rewarding because it provides a much deeper insight into the process. This strategy

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has been used by Enriquez et al. [10] and Collamore, Vidyashankar [7] to obtain an explicit formula for the limiting constant c_+ in (1.4) and also by Buraczewski et al. [4] to prove large deviations results.

1.2. Exceedence times. In this paper we analyze path behavior of the process $\{Y_n\}$ as well as of its maximums

$$(1.5) \quad M_n := \max_{0 \leq k \leq n} Y_k, \quad n = 0, 1, \dots$$

Our primary objective is to study the asymptotic distribution of the first passage time

$$\tau_u := \inf \{n : Y_n > u\} = \inf \{n : M_n > u\} \quad \text{as } u \rightarrow \infty.$$

or of the scaled first passage time

$$T_u = \frac{1}{\log u} \inf \{n : Y_n > u\}.$$

This is a basic question clearly motivated by the large deviation theory for random walks, see the work of Laley [15]. Some partial results in this direction were proved by Nyrhinen [18, 19]. A real break through has been obtained in [3] and the aim of the present paper is to pursue the investigation further.

In [3] we were able to describe the distribution of T_u , which in particular gives the asymptotic of

$$\mathbb{P}[\tau_u \in I_u]$$

for intervals I_u around $\rho \log u$, $\rho > 0$. The length of I_u , depending on the value of ρ , is of the order $\log \log u$ or $\sqrt{\log u}$. This implies that $\log u$ is the correct scaling for perpetuities as it was for the random walks. More details will be given below in subsection 1.6.

In this paper, under some continuity assumption on A , we go a step further and describe the pointwise behavior of τ_u , that is the asymptotic of

$$\mathbb{P}[\tau_u = \lfloor \rho \log u \rfloor], \quad u \rightarrow \infty.$$

The results we obtain are partly with analogy to random walks but partly they are completely different and, may be, this is the most interesting observation we are making here, see Theorems 1.10 and 1.17.

1.3. A class of stochastic recursions. Before stating our main results, we introduce some notation related to the stochastic recursions we have in mind. The sequences $\{Y_n\}$ and $\{M_n\}$ represent the backward processes generated by the random recursions

$$X_n = A_n X_{n-1} + B_n, \quad \text{and} \quad M_n^* = (A_n M_{n-1}^* + B_n, 0)^+,$$

respectively. If $\mathbb{E}[\log A] < 0$ and $\mathbb{E}[\log^+ |B|] < \infty$, then it is well-known that both X_n and M_n^* converge in distribution to Y and $M = \sup_{n \geq 0} Y_n$. This follows from pointwise convergence of $\{Y_n\}$ and M_n ; see Vervaat [22] for more details. Then Y and M are called stationary solutions, since they satisfy the stochastic fixed point equations

$$Y \stackrel{\mathcal{D}}{=} AY + B, \quad Y \text{ independent of } (A, B);$$

$$M \stackrel{\mathcal{D}}{=} (AM + B)^+, \quad M \text{ independent of } (A, B).$$

1.4. The main result-asymptotics of τ_u . While studying perpetuities the main role is played by fluctuations of the random walk $\Pi_n = A_1 \dots A_n$; see [5, 11]. As far as the B - part is concerned, usually only finiteness of some moments is assumed. It is well-known that the properties of the perpetuity sequence Y_n and the sequence of maximums M_n are essentially determined by the moment generating functions

$$\lambda(s) = \mathbb{E}[A^s], \quad \text{and} \quad \Lambda(s) = \log \lambda(s).$$

Let $D_\mu = [0, s_0)$, $s_0 \leq \infty$, be the domain of λ i.e. $s_0 = \sup\{s : \lambda(s) < \infty\}$. We will assume (see (1.8) below) that $s_0 > 0$, thus D_μ is nonempty. Then an easy argument proves that both λ and Λ are C^∞ , convex functions on D_μ .

Throughout the paper we assume that

$$(1.6) \quad \mathbb{E} \log A < 0 \text{ and } \mu_A - \text{the law of } A \text{ is nonlattice.}$$

The nonlattice assumption is required for the Petrov theorem 2.1, which is the core of our proof. The logarithmic moment assumption (together with moment assumption for B stated below) implies that both $\{Y_n\}$ and $\{M_n\}$ converge a.s. to some random variables Y and M . It may happen that the laws of Y and M are degenerated to one point or bounded, e.g. when the action of the pair (A, B) fixes some point $x \in \mathbb{R}$, that is $\mathbb{P}[Ax + B = x] = 1$, then $Y = x$ a.s. However below we exclude this case from our considerations. We assume

$$(1.7) \quad \mathbb{P}[Ax + B = x] < 1 \text{ for every } x \in \mathbb{R}.$$

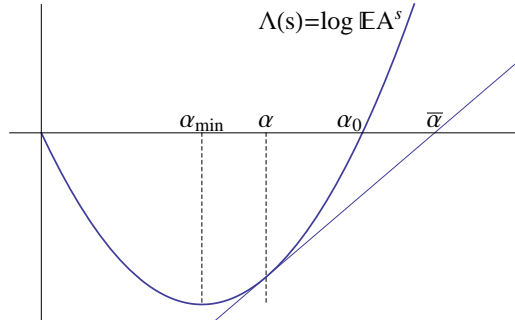
Our main results will be described in terms of a parameter α satisfying

$$(1.8) \quad \Lambda'(\alpha) > 0 \text{ and there is } \xi \text{ such that } \mathbb{E}A^{\alpha+\xi} < \infty, \mathbb{E}|B|^{\alpha+\xi} < \infty.$$

and such α is assumed to exist. In particular, (1.8) implies that $\mathbb{P}[A > 1] > 0$. Then due to a theorem by Guivarc'h-Le Page [13], Y must be unbounded (see also [5]). Also, it follows that there is a unique $\alpha_{\min} \in D_\mu$ such that Λ attains its minimum at α_{\min} . The asymptotic we obtain depends on the parameter $\bar{\alpha}$:

$$(1.9) \quad \bar{\alpha} = \alpha - \frac{\Lambda(\alpha)}{\Lambda'(\alpha)}.$$

$\bar{\alpha}$ has a geometric interpretation. Namely, the tangent line to Λ at point α intersects the x -axis at $\bar{\alpha}$. See the figure below.



Our main result is as follows:

Theorem 1.10. *Assume that (1.6), (1.7) and (1.8) are satisfied and*

(1.11) *either $\alpha_{\min} \leq 1$ or $\Lambda(1) < \Lambda(\alpha)$;*

(1.12) *there are $(a_1, b_1), (a_2, b_2) \in \text{supp } \mu$ such that $a_1 < 1$, $a_2 > 1$ and $\frac{b_2}{1-a_2} < \frac{b_1}{1-a_1}$;*

(1.13) *the law of A has density $f_A(a)$ satisfying $f_A(a) \leq C(1+a)^{-D}$ for some $D > 1+\alpha$;*

(1.14) $\mathbb{P}[A \in da, B \in db] \leq f_A(a)dad\nu(b)$ *for some probability measure ν .*

Then

$$(1.15) \quad \mathbb{P}[\tau_u = \lfloor k_u \rfloor] \sim \frac{C\lambda(\alpha)^{-\Theta(u)}}{\sqrt{\log u}} u^{-\bar{\alpha}},$$

for some strictly positive constant C , $k_u = \frac{\log u}{\Lambda'(\alpha)}$ and $\Theta(u) = k_u - \lfloor k_u \rfloor$.¹

1.5. Some comments on Theorem 1.10. Before proceeding further, let us discuss assumptions of Theorem 1.10 and the result. The general shape of (1.15) matches with what is known about T_u both for random walks [15] and perpetuities [3] as it is explained in subsection 1.6. In particular, we identify the most likely passage time of Y_n into the set (u, ∞) . This happens when the parameter $\bar{\alpha}$ is the smallest, that is when $\alpha = \alpha_0 = \bar{\alpha}$ (see the figure above).

However, now τ_u is much better localized and so the assumptions are considerably stronger than those of [3]:

- (1.11) indicates the optimal set of indices. Indeed, $\alpha_{\min} \leq 1$ then (1.15) holds for every $\alpha > \alpha_{\min}$. If not, we require the condition

$$(1.16) \quad \Lambda(1) < \Lambda(\alpha),$$

which is well justified by Theorem 1.17 that provides a class of appropriate counterexamples to (1.15). If $\alpha_{\min} > 1$ then there is $1 < \tilde{\alpha} < \alpha_0$ such that $\Lambda(1) = \Lambda(\tilde{\alpha})$, (1.16) is satisfied for $\alpha > \tilde{\alpha}$, and we have (1.15) for $k_u < \frac{\log u}{\Lambda'(\tilde{\alpha})}$. If $k_u \geq \frac{\log u}{\Lambda'(\tilde{\alpha})}$ (i.e. $\alpha_{\min} < \alpha \leq \tilde{\alpha}$) the asymptotic may be of different order as we will see below in Theorem 1.17.

- Assumption (1.12) is needed to ensure that both processes Y_n and M_n will exceed with positive probability the level u for an arbitrary large u . Indeed, (1.7) together with $\mathbb{P}[A > 1] > 0$ imply that Y_n is unbounded, but not necessarily at $+\infty$. (1.12) is the weakest assumption implying that Y_n exceed arbitrary high level u with positive probability (see [5]).
- Assumptions (1.13) and (1.14) are technical. The strategy of the proof requires that μ - the joint law of (A, B) is dominated by a product and that the law of A has a density that decays properly at $+\infty$.
- Observe that (1.7) is indeed implied by (1.12).

¹The function $\Theta(u)$ is a correction term, which is needed because τ_u is a discrete time, whereas k_u depends on u in a continuous way.

The Theorem follows in a natural way due to the behavior of Y_n on a different probability space. From the technical point of view, we change the probability measure, which brings the value of the drift adapted to the hitting moment. Suddenly, what was a rare even in the original probability space stops being such in the new one. Uniform large deviation theorem due to Petrov (see Theorem 2.1 below) plays fundamental role here.

Our second main result shows that assumption (1.11) in Theorem 1.10 is indispensable. We consider α such that $\Lambda'(\alpha)$ is close to 0 and the times k_u are large. Then behavior of the passage time may be of different order.

Theorem 1.17. *Assume that (1.6), (1.7) and (1.8) are satisfied $\alpha_{\min} > 1$ and*

$$(1.18) \quad \Lambda(\alpha) < \Lambda(1);$$

$$(1.19) \quad B > 0, \text{ a.s.};$$

$$(1.20) \quad \text{there are } 0 < b_1 < b_2 \text{ and a non vanishing continuous function } f_A \text{ such that}$$

$$\mathbb{P}[A \in da, B_n \in \mathbf{1}_{(b_1, b_2)}(b)db] \geq f_A(a)d\alpha(b).$$

Then there exists $\delta > 0$ such that

$$\mathbb{P}[\tau_u = \lfloor k_u \rfloor] \geq \frac{C}{\sqrt{\log u}} u^{-\bar{\alpha}} u^\delta.$$

Remark 1.21. The assumption (1.20) on f_A can be improved to: $f_A da$ is a measure containing non trivial absolutely continuous part. We comment more on it in Remark 6.7 at the end of Section 6. Similarly as in the previous theorem, (1.7) follows from (1.18) and $\mathbb{P}[A > 1] > 0$.

1.6. Previous results and the analogy with the random walk $S_n = \log A_1 + \dots + \log A_n$. Usually while studying exceedance probabilities for the random walks three cases $\alpha > \alpha_0$, $\alpha = \alpha_0$ and $\alpha < \alpha_0$ are distinguished. As we see have just seen it is not that simple for perpetuities and now we are going to comment on that. In this section assumptions (1.12)-(1.14) are no longer required.

Let us discuss first the case $\alpha > \alpha_0$. Then, as it was proved in [3]

$$(1.22) \quad \mathbb{P}[\tau_u < k_u] = \mathbb{P}[T_u < \Lambda'(\alpha)^{-1}] = \frac{C_1(\alpha)u^{-\bar{\alpha}}}{\sqrt{\log u}} \lambda(\alpha)^{-\Theta(u)}(1 + o(1))$$

with C_1 identified explicitly. Moreover,

$$(1.23) \quad \mathbb{P}[\tau_u < k_u - D \log \log u] = o\left(\frac{u^{-\bar{\alpha}}}{\sqrt{\log u}}\right),$$

for appropriately large constant D . (1.22) is clearly weaker then (1.15). But instead, (1.22) is in full analogy to the Lalley's result (1.25) for the random walks.

Let $\{X_i\} \subset \mathbb{R}$ be an i.i.d. sequence of random variables such that $\mathbb{E}[X_i] < 0$ and let

$$(1.24) \quad S_n = X_1 + \dots + X_n, \quad n \in \mathbb{Z}_+; \quad S_0 = 0.$$

Set

$$\mathcal{T}_u = \frac{1}{u} \inf \{n : S_n \geq u\}, \quad u \geq 0.^2$$

²Now u is the correct scaling

Suppose that the distribution of X_1 is non-lattice, and that there exists a value $\alpha_0 > 0$ such that $\mathbb{E}e^{\alpha_0 X_1} = 1$. Let $\alpha > \alpha_0$, and assume that $\lambda(\beta) = \mathbb{E}e^{\beta X_1}$ is defined in an neighborhood of α .³ Then it is shown in [15], Theorem 5, that

$$(1.25) \quad \mathbb{P}\{\mathcal{T}_u \leq \tau\} = \frac{C_1(\alpha)(\lambda(\alpha))^{-\Theta_1(u)}}{\alpha\sigma(\alpha)\sqrt{2\pi\tau u}} e^{-u\bar{\alpha}} (1 + o(1)) \quad \text{as } u \rightarrow \infty,$$

where $\tau = \Lambda'(\alpha)^{-1}$ and $\Theta_1(u) =: \tau u - \lfloor \tau u \rfloor \in (0, 1)$. Similarly, if $\alpha < \alpha_0$, then

$$(1.26) \quad \mathbb{P}\{\mathcal{T}_u \in (\tau, \infty)\} = \frac{C_2(\alpha)(\lambda(\alpha))^{\Theta_2(u)}}{\alpha\sigma(\alpha)\sqrt{2\pi\tau u}} e^{-u\bar{\alpha}} (1 + o(1)) \quad \text{as } u \rightarrow \infty,$$

where $\Theta_2(u) =: \lfloor \tau u + 1 \rfloor - \tau u \in (0, 1)$. As we prove in Theorem 1.17, if $\alpha < \alpha_0$ then the analogy with the random walk breaks down, because for perpetuities we may have different exponents depending whether (1.11) is satisfied or not. This has been already indicated in [3].

Now let us pass to $\alpha = \alpha_0$ i.e. to the “most likely” first passage time into the set (u, ∞) being $k_u = \frac{\log u}{\Lambda'(\alpha_0)}$. In [3] we described the behavior of T_u around its central tendency as follows. There is a positive constant a such that for any $y \in \mathbb{R}$,

$$(1.27) \quad \mathbb{P}[T_u < \Lambda'(\alpha_0)^{-1} + a(\log u)^{-1/2}y] = C_1 u^{-\alpha_0} \Phi(y) (1 + o(1)),$$

as $u \rightarrow \infty$, where Φ is the standard Normal distribution function and α_0 is the same tail index that appears in (1.4). (1.27) translated to the language of τ_u gives $\mathbb{P}[\tau_u \in I_u]$ for an interval of length $\sqrt{\log u}$ around $k_u = \Lambda'(\alpha_0)^{-1} \log u$.

For the random walk $\{S_n\}$ defined as above

$$(1.28) \quad \mathbb{P}\left\{T_u \leq \Lambda'(\alpha_0)^{-1} + au^{-1/2}y\right\} = \mathcal{C}_M^* e^{-\xi u} (1 + o(1)) \Phi(y),$$

where \mathcal{C}_M^* is the Cramér-Lundberg constant [2] and so the behavior of perpetuities and random walks around their central tendency is the same.

It is remarkable that the products $A_1 \cdots A_n$ play such a dominating role of in the behavior of Y_n and there is almost no influence of B except for the approximation constant.

2. PETROV'S THEOREM

To analyze the behavior of the random walk $\{\Pi_n\}$, the following uniform large deviation theorem, due to Petrov (1965, Theorem 2), will play a key role.

Theorem 2.1 (Petrov). *Assume that the law of $\log A$ is nonlattice and that c satisfies $\mathbb{E}[\log A] < c < s_0$, and suppose that $\delta(n)$ is an arbitrary function satisfying $\lim_{n \rightarrow \infty} \delta(n) = 0$. Then with α chosen such that $\Lambda'(\alpha) = c$, we have that*

$$\begin{aligned} & \mathbb{P}\{\Pi_n > e^{n(c+\gamma_n)}\} \\ &= \frac{1}{\alpha\sigma(\alpha)\sqrt{2\pi n}} \exp\left\{-n\left(\alpha(c+\gamma_n) - \Lambda(\alpha) + \frac{\gamma_n^2}{2\sigma^2(\alpha)}(1 + O(|\gamma_n|))\right)\right\} (1 + o(1)) \end{aligned}$$

as $n \rightarrow \infty$, uniformly with respect to c and γ_n in the range

$$(2.2) \quad \mathbb{E}[\log A] + \epsilon \leq c \leq s_0 - \epsilon \quad \text{and} \quad |\gamma_n| \leq \delta(n),$$

³A slight abuse of notation but if $X_i \equiv \log A_i$ for all i , then this notation agrees with the previous one.

where $\epsilon > 0$.

Remark 2.3. In (2.2), we may have that $s_0 = \infty$ or $\mathbb{E}[\log A] = -\infty$. In these cases, the quantities $\infty - \epsilon$ or $-\infty - \epsilon$ should be interpreted as arbitrary positive, respectively negative, constants.

In fact we will also use some refinements of the last result

Lemma 2.4. *Under the hypotheses of the previous lemma assume additionally that δ_n, j_n satisfy*

$$(2.5) \quad \max \{ \sqrt{n} |\delta_n|, j_n / \sqrt{n} \} \leq \bar{\delta}_n.$$

Then

$$\mathbb{P}[\Pi_{n-j_n} \geq te^{n\delta_n}] = \frac{1}{\alpha\sigma(\alpha)\sqrt{2\pi n}} \cdot t^{-\bar{\alpha}} e^{-\alpha n\delta_n} e^{-j_n\Lambda(\alpha)} (1 + o_1(1)) \quad \text{as } n \rightarrow \infty$$

uniformly for all $\Lambda'(\alpha)$ satisfying $\mathbb{E} \log A_1 + c_1 \leq \Lambda'(\alpha) \leq c_2$ and for all δ_n, j_n satisfying (2.5).

Proof. Let $\bar{n} = (n - j_n)$. We write $te^{n\delta_n} = e^{n\rho + n\delta_n}$ and $n\rho + n\delta_n = (n - j_n)\rho + j_n\rho + n\delta_n = \bar{n}\rho + \bar{n}(\frac{j_n\rho}{\bar{n}} + \frac{n\delta_n}{\bar{n}})$. Now we may apply Theorem 2.1 with $\delta'_n = \frac{j_n\rho}{\bar{n}} + \frac{n\delta_n}{\bar{n}}$ playing the role of δ_n and \bar{n} playing the role of n . Clearly then

$$\bar{n} \left(\frac{j_n\rho}{\bar{n}} + \frac{n\delta_n}{\bar{n}} \right)^2 = O(\bar{\delta}_n).$$

Hence we may write

$$\begin{aligned} \mathbb{P}[A_1 \dots A_{n-j_n} \geq te^{n\delta_n}] &= \frac{1}{\alpha\sigma(\alpha)\sqrt{2\pi n}} \cdot e^{-\rho\bar{n}\bar{\alpha}} e^{-\alpha\bar{n}\delta'_n} (1 + o_1(1)) \\ &= \frac{1}{\alpha\sigma(\alpha)\sqrt{2\pi n}} \cdot e^{-\rho n\bar{\alpha}} e^{-\alpha n\delta_n} e^{-j_n\Lambda(\alpha)} (1 + o_1(1)) \end{aligned}$$

which proves the Lemma. \square

3. LOWER ESTIMATES

In this section we prove lower estimates. Our main result is the following.

Proposition 3.1. *Assume (1.6), (1.8) and (1.12) are satisfied, then there is $\eta > 0$ such that*

$$\mathbb{P}[\tau_u = k_u + 1] = \mathbb{P}[M_{k_u} < u \text{ and } Y_{k_u+1} > u] \geq \frac{\eta}{\sqrt{\log u}} u^{-\bar{\alpha}}$$

for $k_u = \frac{\log u}{\Lambda'(\alpha)}$ and sufficiently large u .

The proof of this Proposition bases on two Lemmas. First we consider the following perpetuity

$$\bar{Y}_n = \sum_{j=1}^n \Pi_{j-1} |B_j|$$

and study the joint distribution of Π_n and \bar{Y}_n as $n \rightarrow \infty$.

Lemma 3.2. *Assume (1.6) and (1.8). For a fixed $r_0 > 0$, there is $s \geq 1$, such that for every $\gamma > 0$, $1 < r \leq r_0$ and $u \geq u(\gamma, r)$*

$$(3.3) \quad \mathbb{P}\{\gamma u < \Pi_{k_u} \leq \gamma r u, \quad \bar{Y}_{k_u} \leq \gamma s u\} \geq \frac{D(r)\gamma^{-\alpha}}{\sqrt{\log u}} u^{-\bar{\alpha}},$$

where $k_u = \frac{\log u}{\Lambda'(\alpha)}$ and the function $D(r) > 0$ for $1 < r \leq r_0$ is increasing.

Remark 3.4. *For given r_0 and s the value of $u(\gamma, r)$ is not uniformly bounded in $\gamma, r \rightarrow 0$. The property that s can be chosen independently of $\gamma > 0$ and $1 < r \leq r_0$ is crucial for our proof of Proposition 3.1.*

Proof. By Theorem 2.1, there exists a constant $D(r)$ such that

$$(3.5) \quad \mathbb{P}\{\gamma u < \Pi_{k_u} \leq \gamma r u\} = \frac{D(r)}{\sqrt{\log u}} \gamma^{-\alpha} u^{-\bar{\alpha}} (1 + o(1)) \quad \text{as } u \rightarrow \infty.$$

Recall $u^{-\bar{\alpha}} = u^{-\alpha} (\mathbb{E} A^\alpha)^{k_u}$. Notice we have

$$\begin{aligned} & \mathbb{P}\{\gamma u \leq \Pi_{k_u} \leq \gamma r u, \quad \bar{Y}_{k_u} \leq \gamma s u\} \\ &= \mathbb{P}\{\gamma u \leq \Pi_{k_u} \leq \gamma r u\} - \mathbb{P}\{\gamma u \leq \Pi_{k_u} \leq \gamma r u, \quad \bar{Y}_{k_u} > \gamma s u\}. \end{aligned}$$

We are going to estimate the second summand in the last expression and prove

$$(3.6) \quad \mathbb{P}\{\gamma u \leq \Pi_{k_u} \leq \gamma r u, \quad \bar{Y}_{k_u} > \gamma s u\} \leq \frac{C\gamma^{-\alpha}}{s^\varepsilon} u^{-\bar{\alpha}} \cdot \left(\frac{D(r)}{\sqrt{\log u}} + \frac{1}{\log u} \right)$$

for $s \geq 1$ and $u \geq u(\gamma, r, s)$. Then we take $s = 1 + (2C)^{\frac{1}{\varepsilon}}$, increase $u(\gamma, r, s)$ if necessarily and conclude the Lemma.

Notice that to prove (3.6), without any loss of generality, we may assume that $|B_k| > 1$ a.s. We write

$$\mathbb{P}\{\gamma u \leq \Pi_{k_u} \leq \gamma r u, \bar{Y}_{k_u} > \gamma s u\} \leq \sum_{i \geq 0} \mathbb{P}\left\{ \gamma u \leq \Pi_{k_u} \leq \gamma r u, \Pi_{k_u-i-1} |B_{k_u-i}| > \frac{\gamma s u}{2(i+1)^2} \right\}$$

We take large K and we divide the sum into two parts depending whether $i > K \log k_u$ or $i \leq K \log k_u$.

Case 1. Suppose that $i > K \log k_u$. We take $\beta < \alpha$ and define $\varepsilon = \alpha - \beta$, $\delta = \lambda(\beta)/\lambda(\alpha) < 1$. Moreover, let $\Pi'_i = A_{k_u-i+1} \cdots A_{k_u}$. We write

$$\begin{aligned}
& \mathbb{P} \left\{ \gamma u \leq \Pi_{k_u} \leq \gamma r u, \quad \Pi_{k_u-i-1} |B_{k_u-i}| > \frac{\gamma s u}{2i^2} \right\} \\
& \leq \sum_{m \geq 0} \mathbb{P} \left\{ \Pi_{k_u} \geq \gamma u, \quad \frac{\gamma s u e^m}{2i^2} \leq \Pi_{k_u-i-1} |B_{k_u-i}| < \frac{\gamma s u e^{m+1}}{2i^2} \right\} \\
& = \sum_{m \geq 0} \mathbb{P} \left\{ \Pi_{k_u-i-1} A_{k_u-i} \Pi'_i \geq \gamma u, \quad \frac{\gamma s u e^m}{2i^2} \leq \Pi_{k_u-i-1} |B_{k_u-i}| < \frac{\gamma s u e^{m+1}}{2i^2} \right\} \\
& \leq \sum_{m \geq 0} \int \mathbb{P} \left\{ \Pi_{k_u-i-1} a \Pi'_i \geq \gamma u, \quad \frac{\gamma s u e^m}{2i^2} \leq \Pi_{k_u-i-1} b < \frac{\gamma s u e^{m+1}}{2i^2} \right\} \mu(da d|b|) \\
& \leq \sum_{m \geq 0} \int \mathbb{P} \left\{ \Pi_{k_u-i-1} \geq \frac{\gamma s u e^m}{2i^2 b} \right\} \cdot \mathbb{P} \left\{ \Pi'_i \geq \frac{2i^2 b}{a s e^{m+1}} \right\} \mu(da d|b|)
\end{aligned}$$

Now we apply twice the Chebyshev inequality and estimate the last expression by

$$\begin{aligned}
& \sum_{m \geq 0} \int \mathbb{P} \left\{ \Pi_{k_u-i-1} \geq \frac{\gamma s u e^m}{2i^2 b} \right\} \cdot \mathbb{P} \left\{ \Pi_i \geq \frac{2i^2 b}{a s e^{m+1}} \right\} \mu(da d|b|) \\
& \leq \sum_{m \geq 0} \int \frac{(2i^2 b)^\alpha}{(\gamma s u e^m)^\alpha} \lambda(\alpha)^{k_u-i} \cdot \frac{(a s e^m)^\beta}{(2i^2 b)^\beta} \lambda(\beta)^i \mu(da d|b|) \\
& \leq \frac{2^{\alpha-\beta}}{\gamma^\alpha s^\varepsilon} \frac{1}{u^\alpha} i^{2\varepsilon} \delta^i \sum_{m \geq 0} e^{-m\varepsilon} \int b^\varepsilon a^\beta \mu(da d|\beta|) \\
& = \frac{C i^{2\varepsilon} \delta^i}{\gamma^\alpha s^\varepsilon} u^{-\bar{\alpha}}
\end{aligned}$$

with the constant C depending only on α, β and μ . Summing over i we obtain

$$\begin{aligned}
& \sum_{i > K \log k_u} \mathbb{P} \left\{ \gamma u \leq \Pi_{k_u} \leq \gamma r u, \quad \Pi_{k_u-i-1} |B_{k_u-i}| > \frac{\gamma s u}{2i^2} \right\} \\
& \leq \sum_{i > K \log k_u} \frac{C i^{2\varepsilon} \delta^i}{\gamma^\alpha s^\varepsilon} u^{-\bar{\alpha}} \leq \frac{C}{s^\varepsilon \gamma^\alpha \log u} u^{-\bar{\alpha}}
\end{aligned}$$

provided K is sufficiently large. Note that we can choose K depending only on μ, α and β .

Case 2. Now we assume that $i \leq K \log k_u$ and that L is large enough and satisfy

$$\begin{aligned}
(3.7) \quad & -\alpha L + 1 < 0 \quad \text{if } \Lambda(\alpha) \geq 0, \\
& -\alpha L + 1 - \Lambda(\alpha)K < 0 \quad \text{if } \Lambda(\alpha) < 0.
\end{aligned}$$

Then

$$\begin{aligned}
 (3.8) \quad & \mathbb{P} \left\{ \gamma u \leq \Pi_{k_u} \leq \gamma r u, \quad \Pi_{k_u-i-1} |B_{k_u-i}| > \frac{\gamma s u}{2(i+1)^2} \right\} \\
 & \leq \mathbb{P} \left\{ \Pi_{k_u-i-1} |B_{k_u-i}| \geq \gamma s u \cdot k_u^L \right\} \\
 & \quad + \mathbb{P} \left\{ \gamma u \leq \Pi_{k_u} \leq \gamma r u, \quad \frac{\gamma s u}{2(i+1)^2} \leq \Pi_{k_u-i-1} |B_{k_u-i}| \leq \gamma s u \cdot k_u^L \right\}
 \end{aligned}$$

The first term is asymptotically negligible, since by the Chebyshev inequality and (3.7) (then the term in the brackets below is uniformly bounded for $i \leq K \log k_u$)

$$\begin{aligned}
 (3.9) \quad & \mathbb{P} \{ \Pi_{k_u-i-1} |B_{k_u-i}| \geq \gamma s u \cdot k_u^L \} \leq \mathbb{P} \{ \Pi_{k_u-i-1} \geq \gamma s u \cdot k_u^L \} \\
 & \leq \frac{1}{\gamma^\alpha s^\alpha u^\alpha k_u^{\alpha L}} \lambda(\alpha)^{k_u-i-1} \mathbb{E}[|B|^\alpha] \\
 & \leq \frac{C \gamma^{-\alpha} s^{-\alpha}}{\log u} u^{-\bar{\alpha}} \left(k_u^{-\alpha L+1} e^{-(i+1)\Lambda(\alpha)} \right) \\
 & \leq \frac{C}{\gamma^\alpha s^\alpha \log u} u^{-\bar{\alpha}}
 \end{aligned}$$

We will use this estimate later on.

Next to estimate the second term in (3.8) we write

$$\begin{aligned}
 & \mathbb{P} \left\{ \gamma u \leq \Pi_{k_u} \leq \gamma r u, \quad \frac{\gamma s u}{2i^2} \leq \Pi_{k_u-i-1} |B_{k_u-i}| \leq \gamma s u \cdot k_u^L \right\} \\
 & = \mathbb{P} \left\{ \gamma u \leq \Pi_{k_u-i-1} A_{k_u-i} \Pi'_i \leq \gamma r u, \quad \frac{\gamma s u}{2i^2} \leq \Pi_{k_u-i-1} |B_{k_u-i}| \leq \gamma s u \cdot k_u^L \right\} \\
 & \leq \sum_{0 \leq m \leq \log(2i^2 k_u^L)} \int \mathbb{P}(U(a, b, m)) \mu(da, d|b|)
 \end{aligned}$$

for

$$U(a, b, m) = \left\{ \gamma u \leq \Pi_{k_u-i-1} a \Pi'_i \leq \gamma r u, \quad \frac{\gamma s u e^m}{2i^2} \leq \Pi_{k_u-i-1} b \leq \frac{\gamma s u \cdot e^{m+1}}{2i^2} \right\}.$$

Now we dominate the sets $U(a, b, m)$ as follows

$$\begin{aligned}
 \mathbb{P}(U(a, b, m)) & \leq \mathbb{P} \left\{ \gamma u \leq \Pi_{k_u-i-1} a \Pi'_i \leq \gamma r u, \quad \frac{2bi^2}{sae^{m+1}} \leq \Pi'_i \leq \frac{2rbi^2}{sae^m} \right\} \\
 & \leq \int \mathbb{P} \left\{ \frac{\gamma u}{aw} \leq \Pi_{k_u-i-1} \leq \frac{\gamma r u}{aw} \right\} \mathbb{P} \left\{ \frac{2bi^2}{sae^{m+1}} \leq \Pi'_i \leq \frac{2rbi^2}{sae^m}, \Pi'_i \in (w, w+dw) \right\} \\
 & \leq \sup_{\frac{2bi^2}{sae^{m+1}} \leq w \leq \frac{2rbi^2}{sae^m}} \mathbb{P} \left\{ \frac{\gamma u}{aw} \leq \Pi_{k_u-i-1} \leq \frac{\gamma r u}{aw} \right\} \mathbb{P} \left\{ \frac{2bi^2}{sae^{m+1}} \leq \Pi'_i \leq \frac{2rbi^2}{sae^m} \right\}
 \end{aligned}$$

Since both m and i are bounded by a constant times $\log k_u$, we can apply Petrov's Theorem 2.1 on the set

$$\Theta = \{b \leq e^{\sqrt{k_u}}\}$$

Case 2a. Applying Petrov's Theorem 2.1 and Chebyshev inequality we have

$$\begin{aligned}
& \sum_{0 \leq m \leq \log(2i^2 k_u^L)} \int \mathbf{1}_{\Theta}(b) \mathbb{P}(U(a, b, m)) \mu(da, d|b|) \\
& \leq \sum_{0 \leq m \leq \log(2i^2 k_u^L)} \int \mathbf{1}_{\Theta}(b) \frac{CD(r)}{\sqrt{k_u - i - 1}} \frac{(i^2 b)^\alpha}{(\gamma s e^m)^\alpha} \frac{\lambda(\alpha)^{k_u - i}}{u^\alpha} \frac{(ase^m)^\beta}{(bi^2)^\beta} \lambda(\beta)^i \mu(da, d|b|) \\
& \leq \frac{CD(r)}{\gamma^\alpha s^\varepsilon} \frac{1}{\sqrt{k_u}} u^{-\bar{\alpha}} i^{2\varepsilon} \delta^i \sum_{m \geq 0} e^{-m\varepsilon} \cdot \int b^\varepsilon a^\beta \mu(da, d|b|) \\
& \leq \frac{CD(r)}{\gamma^\alpha s^\varepsilon \sqrt{k_u}} u^{-\bar{\alpha}}
\end{aligned}$$

with the constant C depending only on α, β and μ .

Case 2b. Applying twice the Chebyshev inequality we obtain

$$\begin{aligned}
& \sum_{0 \leq m \leq \log(2i^2 k_u^L)} \int \mathbf{1}_{\Theta^c}(b) \mathbb{P}(U(a, b, m)) \mu(da, d|b|) \\
& \leq \sum_{0 \leq m \leq \log(2i^2 k_u^L)} \int \mathbf{1}_{\Theta^c}(b) \frac{(2i^2 b)^\alpha}{(\gamma s e^m)^\alpha} \cdot \frac{\lambda(\alpha)^{k_u - i}}{u^\alpha} \cdot \frac{(ase^m)^\beta}{(bi^2)^\beta} \cdot \lambda(\beta)^i \mu(da, d|b|) \\
& \leq \frac{2^\alpha}{\gamma^\alpha s^\varepsilon} u^{-\bar{\alpha}} i^{2\varepsilon} \delta^i \sum_{m \geq 0} e^{-m\varepsilon} \cdot \int \mathbf{1}_{\Theta^c}(b) b^\varepsilon a^\beta \mu(da, d|b|) \\
& \leq \frac{C}{\gamma^\alpha s^\varepsilon} u^{-\bar{\alpha}} \mathbb{E}|B|^\varepsilon A^\beta \mathbf{1}_{\Theta^c}
\end{aligned}$$

We estimate the integral by the Hölder inequality with $p_1 = \frac{\alpha}{\beta}, p_2 = \frac{\alpha}{\varepsilon}, \frac{1}{p_1} + \frac{1}{p_2} = 1$, applied to variables $A^\beta, B^\varepsilon \mathbf{1}_{\Theta^c}$. We get

$$\int \mathbf{1}_{\Theta^c}(b) b^\varepsilon a^\beta \mu(da, d|b|) \leq \mathbb{E}[A^\alpha]^{\frac{\beta}{\alpha}} \mathbb{E}[B|^\alpha \mathbf{1}_{\Theta^c}]^{\frac{\varepsilon}{\alpha}} \leq e^{-\frac{\delta\varepsilon}{\alpha} \sqrt{k_u}} \mathbb{E}[A^\alpha]^{\frac{\beta}{\alpha}} \mathbb{E}[B|^\alpha \mathbf{1}_{\Theta^c}]^{\frac{\varepsilon}{\alpha}} \leq \frac{C}{k_u}$$

The second inequality follows by Chebyshev inequality.

Finally, by (3.9) and above estimates we obtain the estimate in case 2

$$\begin{aligned}
& \sum_{i \leq K \log k_u} \mathbb{P} \left\{ \gamma u \leq \Pi_{k_u} \leq \gamma r u, \Pi_{k_u - i - 1} |B_{k_u - i}| > \frac{\gamma s u}{2(i+1)^2} \right\} \\
& \leq \frac{C}{\gamma^\alpha \sqrt{\log u}} u^{-\bar{\alpha}} \sum_{i \leq K \log k_u} \left(\frac{s^{-\alpha} k_u^{-\alpha L + 1} e^{-i\Lambda(\alpha)}}{\sqrt{\log u}} + 2D(r) s^{-\varepsilon} i^{2\varepsilon} \delta^i \right) \\
& \leq \frac{C}{\gamma^\alpha \log u} u^{-\bar{\alpha}} s^{-\alpha} + \frac{CD(r)}{\gamma^\alpha \sqrt{\log u}} u^{-\bar{\alpha}} s^{-\varepsilon}
\end{aligned}$$

for $s \geq 1$. Combining both cases we obtain (3.6) and the lemma follows. \square

Lemma 3.10. *Suppose that (1.12) is satisfied. Then there is n such that*

$$(3.11) \quad \mathbb{P}[\Pi_n > 1 \text{ and } Y_n > 0 \text{ and } Y_k < Y_n \text{ for } k = 1, \dots, n-1] > 0$$

In particular

$$\mathbb{P}[\Pi_n > 1 \text{ and } Y_n > 0] > 0$$

for some n .

Proof. Here we use assumption (1.12). Of course the lemma holds for $n = 1$ when $b_2 > 0$, hence we assume in the proof $b_2 < 0$. Then $b_1 > 0$. We fix parameters δ, N, M (they values will be specified below). Define

$$U_\delta(a, b) = \{(a', b') : |a' - a| \leq \delta a \text{ and } |b' - b| \leq \delta |b|\}$$

and choose (A_k, B_k) such that

$$(3.12) \quad (A_k, B_k) \in U_\delta(a_2, b_2) \text{ for } k = 1, \dots, N;$$

$$(3.13) \quad (A_k, B_k) \in U_\delta(a_1, b_1) \text{ for } k = N + 1, \dots, N + M,$$

for $\delta < \min\{|b_2|, b_1\}$. By assumption (1.12) the probability of the sets just defined is positive. We consider the perpetuity

$$Y_j = \sum_{i=1}^j A_1 \cdots A_{i-1} B_i = \sum_{i=1}^j \Pi_{i-1} B_i.$$

We have

$$\begin{aligned} Y_{N+M} &= Y_N + \Pi_N \sum_{j=1}^M a_1^{j-1} b_1 \\ &\geq \sum_{j=1}^N ((1 + \delta)a_2)^{j-1} b_2 (1 + \delta) + ((1 - \delta)a_2)^N \sum_{j=1}^M ((1 - \delta)a_1)^{j-1} b_1 (1 - \delta) \\ &= b_2 (1 + \delta) \frac{(1 + \delta)^N a_2^N - 1}{(1 + \delta)a_2 - 1} + (1 - \delta)^N a_2^N \frac{(1 - \delta)^M a_1^M - 1}{(1 - \delta)a_1 - 1} b_1 (1 - \delta). \end{aligned}$$

Denote the last expression by $f(\delta)$. We will find integers N and M such that $a_2^N a_1^M > 1$ and $f(0) > 0$. Then by continuity, there exists $\delta > 0$ such that $f(\delta) > 0$ and simultaneously $\Pi_{N+M} = A_1 \dots A_{N+M} > 1$ for any A_1, \dots, A_{N+M} satisfying (3.12), (3.13). We have

$$f(0) = \frac{b_1}{1 - a_1} (1 - a_1^M) a_2^N + \frac{b_2}{1 - a_2} (1 - a_2^N).$$

To prove that the last expression is strictly positive recall

$$\frac{b_1}{1 - a_1} > \frac{b_2}{1 - a_2} > 0.$$

Since this is strict inequality and $a_1 < 1$ we can take large M such that

$$\frac{b_1}{1 - a_1} (1 - a_1^M) > \frac{b_2}{1 - a_2}.$$

Now, for any $N \geq 1$ we have

$$\frac{b_1}{1 - a_1} (1 - a_1^M) a_2^N > \frac{b_2}{1 - a_2} a_2^N > \frac{b_2}{1 - a_2} (a_2^N - 1)$$

and this imply $f(0) > 0$. We take N large enough to satisfy $a_1^M a_2^N > 1$.

Notice that $Y_j < 0$ for $j \leq N$, hence

$$Y_j < Y_{N+M} \text{ for } j = 1, \dots, N.$$

Moreover since for $j > N$

$$Y_{j+1} - Y_j = \Pi_j B_{j+1} > 0$$

the sequence increases for $j > N$ and attains its maximum for $j = N + M$. Therefore also

$$Y_j < Y_{N+M} \text{ for } j = N + 1, \dots, N + M - 1.$$

Finally since

$$\begin{aligned} \mathbb{P}[\Pi_{N+M} > 1 \text{ and } Y_{N+M} > 0 \text{ and } Y_k < Y_{N+M} \text{ for } k = 1, \dots, N + M - 1] \\ \geq \mathbb{P}[(A_k, B_k) \in U_\delta(a_2, b_2) \text{ for } k = 1, \dots, N; \\ (A_k, B_k) \in U_\delta(a_1, b_1) \text{ for } k = N + 1, \dots, N + M] > 0 \end{aligned}$$

we conclude the lemma for $n = N + M$. \square

Proof of Proposition 3.1. We will first prove the result under an additional assumption that $\mathbb{P}[A > 1, B > 0] > 0$. Fix a point $(C_A, C_B) \in \text{supp}\mu$ such that $C_A > 1$ and $C_B > 0$. Define

$$\theta = \left(1 - \frac{3}{4}\varepsilon_0\right) \left(\frac{C_B}{C_A^j} \frac{C_A^j - 1}{C_A - 1}\right)^{-1},$$

Notice that for all $0 < \varepsilon_0 \leq 1$ and $j \geq 2$ we have the inequalities $\theta_1 \geq \theta \geq \theta_0 > 0$, where constants θ_0, θ_1 depends on C_A, C_B only. Fix any $\varepsilon_0 \leq \frac{C_A - 1}{4C_A} \leq 1$ satisfying

$$\theta_0 C_B > 2\varepsilon_0.$$

Let s_0 satisfies the conclusion of Lemma 3.2 for $r_0 = 4$. We put $C_2 = s_0 \theta_1$ and fix large enough j such that

$$\frac{C_2}{C_A^j} < \frac{\varepsilon_0}{2}$$

For very small $0 < \delta \leq \frac{1}{2}$ (that will be defined slightly later) consider the set

$$\Omega = \left\{ \frac{\theta u(1 - \delta)}{C_A^j} < \Pi_{k_u - j} < \frac{\theta u(1 + \delta)}{C_A^j}, \bar{Y}_{k_u - j} < \frac{C_2 u}{C_A^j}, \right. \\ \left. (A'_k, B'_k) \in U_\delta(C_A, C_B) \text{ for } k = 1, \dots, j + 1 \right\}$$

Notice that on the set Ω we have

$$\begin{aligned}
\Pi_{k_u-j} Y'_j &= \Pi_{k_u-j} (B'_1 + A'_1 B'_2 + \cdots + A'_1 \cdots A'_{j-1} B'_j) \\
&\leq \frac{\theta u(1+\delta)}{C_A^j} \cdot (1+\delta) C_B \cdot \frac{C_A^j(1+\delta)^j - 1}{C_A(1+\delta) - 1} \\
&= D_1(\delta) u \\
\Pi_{k_u-j} Y'_j &\geq \frac{\theta u(1-\delta)}{C_A^j} \cdot (1-\delta) C_B \cdot \frac{C_A^j(1-\delta)^j - 1}{C_A(1-\delta) - 1} \\
&= D_2(\delta) u \\
\Pi_{k_u} B_{k_u+1} &\geq \frac{\theta u(1-\delta)}{C_A^j} \cdot C_A^j (1-\delta)^j \cdot C_B (1-\delta) = \theta C_B (1-\delta)^{j+2} \\
&= D_3(\delta) u.
\end{aligned}$$

moreover, by direct computation

$$\begin{aligned}
D_1(0) &= D_2(0) = \frac{\theta C_B}{C_A^j} \cdot \frac{C_A^j - 1}{C_A - 1} = 1 - \frac{3}{4} \varepsilon_0, \\
D_3(0) &= \theta C_B \geq \theta_0 C_B \geq 2 \varepsilon_0.
\end{aligned}$$

Therefore, by continuity of $D_i(\delta)$ one can choose $\delta \leq \frac{1}{2}$ such that

$$D_1(\delta) < 1 - \frac{\varepsilon_0}{2}, \quad D_2(\delta) > 1 - \varepsilon_0, \quad D_3(\delta) \geq \frac{3}{2} \varepsilon_0.$$

Hence we have

$$\begin{aligned}
\Omega &\subset \left\{ \Pi_{k_u-j} Y'_{j-1} \in ((1-\varepsilon_0)u, (1-\varepsilon_0/2)u), \bar{Y}_{k_u-j} < \frac{\varepsilon_0}{2} u, \Pi_{k_u} B_{k_u+1} \geq (3/2)\varepsilon_0 u, M_{k_u} < u \right\} \\
&\subset \left\{ Y_{k_u} \in ((1-3\varepsilon_0/2)u, u), \Pi_{k_u} B_{k_u+1} \geq 3\varepsilon_0 u/2, M_{k_u} < u \right\} \\
&\subset \left\{ M_{k_u} < u \text{ and } Y_{k_u+1} > u \right\},
\end{aligned}$$

On the other hand by Lemma 3.2 applied with

$$\gamma = \frac{\theta(1-\delta)}{C_A^j}, \quad r = \frac{1+\delta}{1-\delta}, \quad r_0 = 4, \quad \text{and} \quad s = \frac{C_2}{\theta(1-\delta)} \geq s_0 = \frac{C_2}{\theta_1}$$

we obtain

$$\begin{aligned}
\mathbb{P}(\Omega) &= \mathbb{P} \left\{ \frac{\theta u(1-\delta)}{C_A^j} < \Pi_{k_u-j} < \frac{\theta u(1+\delta)}{C_A^j}, \bar{Y}_{k_u-j} < \frac{C_2 u}{C_A^j} \right\} \\
&\quad \cdot \mathbb{P} \{ (A'_k, B'_k) \in U_\delta(C_A, C_B) \text{ for } k = 1, \dots, j+1 \} \geq \frac{\eta}{\sqrt{\log u}} u^{-\bar{\alpha}}
\end{aligned}$$

for some very small constant η .

If $\mathbb{P}[A > 1, B > 0] = 0$ we apply Lemma 3.10 and proceed as above. This time we fix a point $(C_A, C_B) \in \text{supp} \mu^{*n}$ such that $C_A > 1$ and $C_B > 0$, but instead of choosing (A'_k, B'_k)

with the law μ close to (C_A, C_B) one has to pick up $(\tilde{A}_k, \tilde{B}_k)$ with the law μ^{*n} (i.e. partial products and perpetuities). This means:

$$\begin{aligned}\tilde{A}_1 &= A'_1 \cdots A'_n, \quad \tilde{B}_1 = \sum_{i=1}^n A'_1 \cdots A'_{i-1} B'_i \\ \tilde{A}_k &= A'_{(k-1)n+1} \cdots A'_{kn}, \quad \tilde{B}_k = \sum_{i=(k-1)n+1}^{kn} A'_{(k-1)n+1} \cdots A'_{i-1} B'_i\end{aligned}$$

and $(\tilde{A}_k, \tilde{B}_k)$ are chosen accordingly. Exactly the same calculations as above give the result. The condition $\{Y_k < Y_n, k < n\}$ in (3.11) is needed, to ensure that the perpetuity will not exceed the level u before time k_u . We omit the details. \square

4. UPPER ESTIMATES

In this section we prove the following result, which gives upper estimates in Theorem 1.10

Proposition 4.1. *Assume that (1.6), (1.8), (1.11), (1.13) and (1.14) are satisfied. Then there exists C such that for every $u \geq 2$ we have*

$$\mathbb{P}[M_{k_u} < u, Y_{k_u+1} > u] \leq \frac{C}{\sqrt{\log u}} u^{-\bar{\alpha}}$$

Moreover there is $\sigma < 1$ and $C > 0$ such that for every $\varepsilon > 0$ and $u \geq 2$ we have

$$(4.2) \quad \mathbb{P}[\tau_u = k_u + 1] \leq \frac{C(\varepsilon^{1-\sigma} + u^{-\xi'})}{\sqrt{\log u}} u^{-\bar{\alpha}}.$$

First we need technical lemmas.

Lemma 4.3. *Assume that (1.6) and (1.8) are satisfied. For any fixed (small) $\delta > 0$ there exist C_δ such that for every $n \in \mathbb{Z}$, $\varepsilon > 0$ and $u \geq 2$ we have*

$$(4.4) \quad \mathbb{P}\left[\Pi_{k_u-1} B_{k_u} > \frac{\varepsilon u}{e^n}\right] \leq \frac{C_\delta e^{\alpha n + \delta|n|}}{\varepsilon^\alpha \sqrt{\log u}} u^{-\bar{\alpha}}$$

Remark 4.5. *The above formula is meaningful only if the right hand side is smaller than 1 but it is useful to write the estimate in the unified way.*

Proof. We consider three cases making distinction depending on whether $n, \log |B_{k_u}|$ are bigger or smaller than $\sqrt{k_u}$.

Case 1. Let first $n > \sqrt{k_u}$. Then by the Chebychev inequality with exponent α we have

$$\begin{aligned}\mathbb{P}\left[\Pi_{k_u-1} B_{k_u} > \frac{\varepsilon u}{e^n}\right] &\leq C e^{k_u \Lambda(\alpha)} \mathbb{E}|B_{k_u}|^\alpha \varepsilon^{-\alpha} u^{-\alpha} e^{\alpha n} \\ &\leq C \varepsilon^{-\alpha} e^{\alpha n + \delta|n|} e^{-\sqrt{k_u} u} u^{-\bar{\alpha}}\end{aligned}$$

and (4.4) follows.

Case 2. If $\log |B_{k_u}| > \sqrt{k_u}$, we write

$$\begin{aligned} \mathbb{P}\left[\Pi_{k_u-1}B_{k_u} > \frac{\varepsilon u}{e^n}\right] &\leq \sum_{m \geq \sqrt{k_u}} \mathbb{P}\left[\Pi_{k_u-1} > \frac{\varepsilon u}{e^{n+m+1}}\right] \mathbb{P}\left[B_{k_u} > e^m\right] \\ &\leq C e^{k_u \Lambda(\alpha)} \varepsilon^{-\alpha} u^{-\alpha} e^{\alpha n} \mathbb{E}|B_{k_u}|^\alpha \sum_{m \geq \sqrt{k_u}} e^{-\delta m} \\ &\leq C \varepsilon^{-\alpha} e^{\alpha n} e^{-\delta \sqrt{k_u}} u^{-\bar{\alpha}} \end{aligned}$$

which gives (4.4).

Case 3. If $\log |B_{k_u}| \leq \sqrt{k_u}$ and $n \leq \sqrt{k_u}$, we use Petrov theorem and we obtain

$$(4.6) \quad \mathbb{P}\left[\Pi_{k_u-1}B_{k_u} > \frac{\varepsilon u}{e^n}\right] \leq C \frac{u^{-\bar{\alpha}}}{\sqrt{\log u}} \varepsilon^{-\alpha} e^{\alpha n} \mathbb{E}|B_{k_u}|^\alpha,$$

which completes the proof. \square

Let $B_n^+ = \max\{B_n, 0\}$ and $Y_n^+ = \sum_{k=1}^n \Pi_{k-1} B_k^+$. Then of course $Y_n^+ \geq Y_n$.

Lemma 4.7. *Assume that (1.6), (1.8) and (1.11) are satisfied. For any fixed (small) $\delta > 0$ there exist C_δ and $\theta < 1$ such that for every $n \in \mathbb{Z}$, $\varepsilon > 0$, $c \geq 1$ and $u \geq 2$ we have*

$$\mathbb{P}\left[\Pi_{k_u-1}B_{k_u} > \frac{\varepsilon u}{e^n} \quad \text{and} \quad Y_{k_u-1}^+ > \frac{cu}{e^n}\right] \leq \frac{C_\delta}{\varepsilon^\theta} \frac{e^{\alpha n + \delta|n|}}{\sqrt{\log u}} u^{-\bar{\alpha}}$$

Remark 4.8. *The above formula is meaningful only if the right hand side is smaller than 1 but it is useful to write the estimate in the unified way. The Lemma will be used with fixed c . The condition $c \geq 1$ can be, of course, replaced by $c \geq c_0 > 0$.*

Proof. Denote

$$g(n, m) = \mathbb{P}\left[\Pi_{k_u-1}B_{k_u} > \frac{\varepsilon u}{e^n} \quad \text{and} \quad \frac{cue^m}{e^n} < Y_{k_u-1}^+ \leq \frac{cue^{m+1}}{e^n}\right]$$

It is sufficient to prove that for $m \geq 0$, $n \in \mathbb{Z}$ and $u \geq \text{const}$

$$(4.9) \quad g(n, m) \leq \frac{C e^{-\delta m}}{\varepsilon^\theta} \frac{e^{\alpha n + \delta|n|}}{\sqrt{\log u}} u^{-\bar{\alpha}}$$

Indeed, then

$$\begin{aligned} \mathbb{P}\left[\Pi_{k_u-1}B_{k_u} > \frac{\varepsilon u}{e^n} \quad \text{and} \quad Y_{k_u-1}^+ > \frac{cu}{e^n}\right] &= \sum_{m \geq 0} g(n, m) \\ &\leq \sum_{m \geq 0} \frac{C e^{-\delta m}}{\varepsilon^\theta} \frac{e^{\alpha n + \delta|n|}}{\sqrt{\log u}} u^{-\bar{\alpha}} \\ &\leq \frac{C}{\varepsilon^\theta} \frac{e^{\alpha n + \delta|n|}}{\sqrt{\log u}} u^{-\bar{\alpha}} \end{aligned}$$

To estimate $g(n, m)$, for $j \geq 1$ we define the set of indices

$$W_j^u = \left\{ 1 \leq i < k_u : \frac{cue^m}{e^n e^j} \leq \Pi_{i-1} B_i^+ \leq \frac{cue^m}{e^n e^{j-1}} \right\}$$

On the set $\left\{\frac{cue^m}{e^n} \leq Y_{k_u-1}^+\right\}$ there is some $j > 0$, such that the number of elements in the set W_j^u must be greater than $\frac{e^j}{10j^2}$. Indeed, assume that such a j does not exist, i.e. for every $j > 0$, $\#W_j^u \leq \frac{e^j}{10j^2}$, then

$$\begin{aligned} Y_{k_u-1}^+ &= \sum_{i < k_u} \Pi_{i-1} B_i^+ = \sum_{j > 0} \sum_{i \in W_j} \Pi_{i-1} B_i^+ \\ &\leq \sum_j \frac{e^j}{10j^2} \cdot \frac{cue^m}{e^n e^{j-1}} < \frac{cue^m}{e^n} \sum_j \frac{e}{10j^2} \\ &< \frac{cue^m}{e^n} \end{aligned}$$

Let

$$K^u = \left\{ (j, m_1, m_2) : j \geq 1, 1 \leq m_1 < k_u, m_1 + \frac{e^j}{10j^2} < m_2 < k_u \right\}.$$

Then

$$(4.10) \quad g(n, m) \leq \sum_{(j, m_1, m_2) \in K^u} \mathbb{P}[U(j, m_1, m_2)],$$

for

$$U(j, m_1, m_2) = \left\{ \frac{cue^m}{e^n e^j} \leq \Pi_{m_i-1} B_{m_i}^+ \leq \frac{cue^m}{e^n e^{j-1}}, \quad i = 1, 2, \quad \text{and} \quad \Pi_{k_u-1} B_{k_u} > \frac{\varepsilon u}{e^n} \right\}.$$

Below we use the convention Π_k, Π'_n, Π''_m to denote independent products of A_j 's of length k, n, m , respectively. Then for any triple $(j, m_1, m_2) \in K^u$ we have

$$\begin{aligned} &\mathbb{P}[U(j, m_1, m_2)] \\ &\leq \int \mathbb{P} \left[\frac{cue^m}{e^n e^j} \leq \Pi_{m_1-1} b_1 \leq \frac{cue^m}{e^n e^{j-1}}, \quad \frac{cue^m}{e^n e^j} \leq \Pi_{m_1-1} a_1 \Pi'_{m_2-m_1} b_2 \leq \frac{cue^m}{e^n e^{j-1}}, \right. \\ &\quad \left. \Pi_{m_1-1} a_1 \Pi'_{m_2-m_1} a_2 \Pi''_{k_u-m_2} b > \frac{\varepsilon u}{e^n} \right] \mathbf{1}_{\{b_1 > 0, b_2 > 0, b > 0\}} \mu(da_1, db_1) \mu(da_2, db_2) \mu(da, db) \\ &\leq \int \mathbb{P} \left[\Pi_{m_1-1} \geq \frac{cue^m}{b_1 e^n e^j} \right] \cdot \mathbb{P} \left[\Pi_{m_2-m_1} > \frac{b_1}{b_2 a_1 e} \right] \cdot \mathbb{P} \left[\Pi_{k_u-m_2} > \frac{\varepsilon e^{j-1} b_2}{c b a_2 e^m} \right] \\ &\quad \cdot \mathbf{1}_{\{b_1 > 0, b_2 > 0, b > 0\}} \mu(da_1, db_1) \mu(da_2, db_2) \mu(da, db) \end{aligned}$$

Fix parameters: $\beta_1 = \alpha - \varepsilon_1, \beta_2 = \alpha - \varepsilon_2$ such that $\beta_1, \beta_2 < 1, \rho_1 = \frac{\lambda(\beta_1)}{\lambda(\alpha)} < 1, \rho_2 = \frac{\lambda(\beta_2)}{\lambda(\alpha)} < 1$ and $\rho_1 > \rho_2$. Here we use (1.11). If $\alpha_{\min} < 1, \Lambda'(\alpha) > 0$ then we take $\alpha_{\min} < \beta_2 < \beta_1 < \min\{1, \alpha\}$. If $\alpha_{\min} \geq 1$ and $\lambda(1) < \lambda(\alpha)$ then there is $\tilde{\alpha} < 1$ such that $\lambda(\tilde{\alpha}) = \lambda(\alpha)$ and so we can take $\beta_1 = \tilde{\alpha} + \varepsilon_1, \beta_2 = \tilde{\alpha} + \varepsilon_2$. We apply the Chebyshev inequality with parameters

α, β_1, β_2 and so

$$\begin{aligned}
(4.11) \quad & \mathbb{P}[U(j, m_1, m_2)] \\
& \leq \int \frac{(b_1 e^n e^j)^\alpha}{c^\alpha u^\alpha e^{\alpha m}} \lambda(\alpha)^{m_1} \cdot \frac{(b_2 a_1 e)^{\beta_1}}{b_1^{\beta_1}} \lambda(\beta_1)^{m_2 - m_1} \frac{(c b a_2 e^m)^{\beta_2}}{(\varepsilon e^{j-1} b_2)^{\beta_2}} \lambda(\beta_2)^{k_u - m_2} \\
& \quad \cdot \mathbf{1}_{\{b_1 > 0, b_2 > 0, b > 0\}} \mu(da_1, db_1) \mu(da_2, db_2) \mu(da, db) \\
& \leq C \varepsilon^{-\beta_2} e^{\alpha n} e^{-\varepsilon_2 m} u^{-\bar{\alpha}} e^{j \varepsilon_2} \rho_1^{m_2 - m_1} \rho_2^{k_u - m_2} \\
& \quad \cdot \int b_1^{\varepsilon_1} b_2^{\beta_1 - \beta_2} a_1^{\beta_1} a_2^{\beta_2} b^{\beta_2} \mathbf{1}_{\{b_1 > 0, b_2 > 0, b > 0\}} \mu(da_1, db_1) \mu(da_2, db_2) \mu(da, db) \\
& \leq C \varepsilon^{-\beta_2} e^{\alpha n} e^{-\varepsilon_2 m} u^{-\bar{\alpha}} e^{j \varepsilon_2} \rho_1^{m_2 - m_1} \rho_2^{k_u - m_2} \mathbb{E}[|B_1|^{\varepsilon_1} A_1^{\beta_1}] \mathbb{E}[|B_2|^{\beta_1 - \beta_2} A_2^{\beta_2}] \mathbb{E}[|B|^{\beta_2}]
\end{aligned}$$

The product of expectations is finite, because of the Hölder inequality and (1.8). Hence it is sufficient to estimate

$$\varepsilon^{-\beta_2} e^{\alpha n} e^{-\varepsilon_2 m} u^{-\bar{\alpha}} \sum_{(j, m_1, m_2) \in K^u} e^{j \varepsilon_2} \rho_1^{m_2 - m_1} \rho_2^{k_u - m_2}.$$

Notice that the sum (of geometric squence) above is always dominated by its maximal term, that is by $C k_u^{2\varepsilon_2} < C k_u$. Assume first that $n > \sqrt{k_u}$. Then combining (4.10) with the estimates above

$$\begin{aligned}
(4.12) \quad g(n, m) & \leq \sum_{(j, m_1, m_2) \in K^u} \mathbb{P}[U(j, m_1, m_2)] \\
& \leq \varepsilon^{-\beta_2} e^{\alpha n} e^{-\varepsilon_2 m} u^{-\bar{\alpha}} k_u^2 \\
& \leq \varepsilon^{-\beta_2} e^{(\alpha + \delta)n} e^{-\delta \sqrt{k_u}} e^{-\varepsilon_2 m} u^{-\bar{\alpha}} k_u^2 \\
& \leq \varepsilon^{-\beta_2} e^{(\alpha + \delta)n} e^{-\varepsilon_2 m} o\left(\frac{u^{-\bar{\alpha}}}{\sqrt{\log u}}\right)
\end{aligned}$$

For the rest we fix C_1 , we assume that $n \leq \sqrt{k_u}$ and we consider 4 cases

$$\begin{aligned}
K_1^u &= \left\{ (j, m_1, m_2) \in K^u : e^{j/2} > C_1 \log k_u \right\}, \\
K_2^u &= \left\{ (j, m_1, m_2) \in K^u : e^{j/2} \leq C_1 \log k_u, m_2 < k_u - k_u^{\frac{1}{4}} \right\}, \\
K_3^u &= \left\{ (j, m_1, m_2) \in K^u : e^{j/2} \leq C_1 \log k_u, m_1 < k_u - 2k_u^{\frac{1}{4}}, m_2 \geq k_u - k_u^{\frac{1}{4}} \right\}, \\
K_4^u &= \left\{ (j, m_1, m_2) \in K^u : e^{j/2} \leq C_1 \log k_u, m_1 \geq k_u - 2k_u^{\frac{1}{4}} \right\}.
\end{aligned}$$

Case 1. In this case there is C_2 such that $m_2 - m_1 > \frac{\varepsilon^j}{10j^2} \geq 2C_2 \log k_u$. Hence

$$\begin{aligned}
g(n, m) & \leq C \varepsilon^{-\beta_2} e^{\alpha n} e^{-\varepsilon_2 m} u^{-\bar{\alpha}} \sum_{(j, m_1, m_2) \in K_1^u} e^{j \varepsilon_2} \rho_1^{m_2 - m_1} \rho_2^{k_u - m_2} \\
& \leq C \varepsilon^{-\beta_2} e^{\alpha n} e^{-\varepsilon_2 m} u^{-\bar{\alpha}} k_u \rho_1^{C_2 \log k_u} \\
& = \varepsilon^{-\beta_2} e^{\alpha n} e^{-\varepsilon_2 m} o\left(\frac{u^{-\bar{\alpha}}}{\sqrt{\log u}}\right)
\end{aligned}$$

Case 2. For the sum over K_2^u we write

$$\begin{aligned} g(n, m) &\leq C\varepsilon^{-\beta_2} e^{\alpha n} e^{-\varepsilon_2 m} u^{-\bar{\alpha}} \sum_{(j, m_1, m_2) \in K_2^u} e^{j\varepsilon_2} \rho_1^{m_2 - m_1} \rho_2^{k_u - m_2} \\ &\leq C\varepsilon^{-\beta_2} e^{\alpha n} e^{-\varepsilon_2 m} u^{-\bar{\alpha}} k_u (\log k_u)^{2\varepsilon_2} \rho_2^{\frac{1}{4}} \\ &= \varepsilon^{-\beta_2} e^{\alpha n} e^{-\varepsilon_2 m} o\left(\frac{u^{-\bar{\alpha}}}{\sqrt{\log u}}\right) \end{aligned}$$

Case 3. Here $m_2 - m_1 > k_u^{\frac{1}{4}}$ and reasoning as above

$$\begin{aligned} g(n, m) &\leq C\varepsilon^{-\beta_2} e^{\alpha n} e^{-\varepsilon_2 m} u^{-\bar{\alpha}} \sum_{(j, m_1, m_2) \in K_3^u} e^{j\varepsilon_2} \rho_1^{m_2 - m_1} \rho_2^{k_u - m_2} \\ &\leq C\varepsilon^{-\beta_2} e^{\alpha n} e^{-\varepsilon_2 m} u^{-\bar{\alpha}} k_u^2 (\log k_u)^{2\varepsilon_2} \rho_1^{\frac{1}{4}} \\ &= \varepsilon^{-\beta_2} e^{\alpha n} e^{-\varepsilon_2 m} o\left(\frac{u^{-\bar{\alpha}}}{\sqrt{\log u}}\right) \end{aligned}$$

Case 4a. On the set $\Theta = \{b_1 \leq e^{\sqrt{k_u}}\}$ and $|n - m| \leq \sqrt{k_u}$ we estimate in a slightly different way the first term, that is

$$\mathbb{P}\left[\Pi_{m_1} \geq \frac{c u e^m}{b_1 e^n e^j}\right].$$

We use Petrov's theorem 2.1 which gives

$$\mathbb{P}\left[\Pi_{m_1} \geq \frac{c u e^m}{b_1 e^n e^j}\right] \leq \frac{C(b_1 e^n e^j)^\alpha}{c^\alpha u^\alpha e^{m\alpha}} \cdot \frac{\lambda(\alpha)^{m_1}}{\sqrt{m_1}}.$$

Then, if $|n - m| \leq 2\sqrt{k_u}$

$$\begin{aligned} \mathbb{P}\left[\Pi_{k_u-1} B_{k_u} > \frac{\varepsilon u}{e^n} \text{ and } \frac{c u e^m}{e^n} < Y_{k_u-1}^+ < \frac{c u e^{m+1}}{e^n}, B \in \Theta\right] \\ \leq C\varepsilon^{-\beta_2} e^{\alpha n} \frac{u^{-\bar{\alpha}}}{\sqrt{\log u}} e^{-\varepsilon_2 m} \sum_{(j, m_1, m_2) \in K_4^u} e^{j\varepsilon} \rho_1^{m_2 - m_1} \rho_2^{k_u - m_2}. \end{aligned}$$

Since, by the definition of K^u the last sum is bounded, we obtain required estimates.

Case 4b. If $m > k_u^{\frac{1}{4}}$ we use (4.11) and obtain

$$g(n, m) \leq C\varepsilon^{-\beta_2} e^{\alpha n} e^{-\varepsilon_2 m/2} o\left(\frac{u^{-\bar{\alpha}}}{\sqrt{\log u}}\right).$$

Case 4c. If $n \leq -\sqrt{k_u}$ and $m < k_u^{\frac{1}{4}}$, then for large u , $e^m \leq e^{\frac{\delta|n|}{2}}$ and hence

$$g(n, m) \leq \mathbb{P}\left[\Pi_{k_u-1} B_{k_u} > \frac{\varepsilon u}{e^n}\right] \leq \frac{e^{-\varepsilon_1 m} e^{\alpha n + \delta|n|}}{\varepsilon^\theta} \frac{u^{-\bar{\alpha}}}{\sqrt{\log u}}.$$

Case 4d. On the set $b_1 \in \Theta^c$ we can sharpen (4.11). We get an extra decay for corresponding expectation $\int \mathbf{1}_{\Theta^c}(b_1) b_1^{\varepsilon_1} a_1^{\beta_1} \mu(da_1, db_1)$.

We use the Hölder inequality with $\frac{1}{p_1} + \frac{1}{p_2} = 1$, p_1 close to 1. Then

$$\int \mathbf{1}_{\Theta^c}(b) b_1^{\varepsilon_1} a_1^{\beta_1} \mu(da_1, db_1) \leq \mathbb{E}[A^{p_1 \beta_1} |B|^{p_1 \varepsilon_1}]^{\frac{1}{p_1}} \cdot \mathbb{P}[|B| > e^{\sqrt{k_u}}]^{\frac{1}{p_2}}$$

and the first term is finite again by the Hölder inequality. Moreover, by (1.8).

$$\mathbb{P}[|B| > e^{\sqrt{k_u}}]^{\frac{1}{p_2}} \leq (\mathbb{E}|B|^\alpha) e^{-\sqrt{k_u} \alpha / p_2}$$

and we deduce as above.

Combining all the cases we obtain (4.9) and complete proof of the Lemma. \square

Proof of Proposition 4.1. We are going to show that for every $0 < \varepsilon < 1$

$$(4.13) \quad \mathbb{P}[Y_{k_u} \in ((1-\varepsilon)u, (1-\varepsilon/2)u), Y_{k_u+1} > u] \leq \varepsilon^{1-\sigma} \frac{Cu^{-\bar{\alpha}}}{\sqrt{\log u}}.$$

(4.13) implies (4.2). Moreover, applying (4.13) to $\varepsilon = 2^{-n}$, $n = 0, 1, 2, \dots$ and summing up over n we obtain Proposition 4.1. Let $J_\varepsilon = (1-\varepsilon, 1-\varepsilon/2)u$. We write $Y_{k_u} = B_1 + A_1 Y'_{k_u-1}$ where $Y'_{k_u-1} = B_2 + A_2 B_3 + \dots + A_2 \dots A_{k_u-1} B_{k_u}$. We will also use notation $\Pi'_{k_u-1} = A_2 \dots A_{k_u}$. We have

$$\mathbb{P}[Y_{k_u} \in J_\varepsilon, \Pi_{k_u} B_{k_u+1} > \varepsilon u] = \mathbb{P}[B_1 + A_1 Y'_{k_u-1} \in J_\varepsilon, \Pi_{k_u} B_{k_u+1} > \varepsilon u]$$

and

$$(4.14) \quad \begin{aligned} & \mathbb{P}[B_1 + A_1 Y'_{k_u-1} \in J_\varepsilon, \Pi_{k_u} B_{k_u+1} > \varepsilon u] \\ & \leq \sum_{n \in \mathbb{Z}} \mathbb{P}\left[e^n \leq A_1 < e^{n+1}, B_1 + A_1 Y'_{k_u-1} \in J_\varepsilon \text{ and } \Pi'_{k_u-1} B_{k_u+1} > \frac{\varepsilon u}{e^{n+1}}\right] \\ & \leq \sum_{n \in \mathbb{Z}} \int \mathbf{1}_{\{\frac{u}{3e^{n+1}} < s < \frac{3u}{e^n}\}} \mathbb{P}\left[e^n \leq A_1 < e^{n+1}, A_1 \in \frac{1}{s}(J_\varepsilon - B_1)\right] \\ & \quad \cdot \mathbb{P}\left[\Pi'_{k_u-1} B_{k_u} > \frac{\varepsilon u}{e^{n+1}}, Y'_{k_u-1} \in ds\right] \end{aligned}$$

Notice that for $s \in (\frac{u}{3e^{n+1}}, \frac{3u}{e^n})$ the interval $\frac{1}{s}(J_\varepsilon - B_1)$ has length at most $\frac{3}{2}\varepsilon e^n$. Thus, by Lemma 4.7

$$\begin{aligned} & \mathbb{P}[B_1 + A_1 Y'_{k_u-1} \in J_\varepsilon, \Pi_{k_u} B_{k_u+1} > \varepsilon u] \\ & \leq \sum_{n \in \mathbb{Z}} \int \mathbf{1}_{\{\frac{u}{3e^{n+1}} < s < \frac{3u}{e^n}\}} C\varepsilon e^n \cdot \min\{e^{-n}, 1\}^D \cdot \mathbb{P}\left[\Pi_{k_u-1} B_{k_u} > \frac{\varepsilon u}{e^{n+1}}, Y_{k_u-1} \in ds\right] \\ & \leq \sum_{n \in \mathbb{Z}} C\varepsilon e^n \cdot \min\{e^{-n}, 1\}^D \cdot \mathbb{P}\left[\Pi_{k_u-1} B_{k_u} > \frac{\varepsilon u}{e^{n+1}}, \frac{u}{3e^{n+2}} < Y_{k_u-1} < \frac{3u}{e^n}\right] \\ & \leq \sum_{n \in \mathbb{Z}} C\varepsilon e^n \cdot \min\{e^{-n}, 1\}^D \cdot \mathbb{P}\left[\Pi_{k_u-1} B_{k_u} > \frac{\varepsilon u}{e^{n+1}}, Y_{k_u-1}^+ > \frac{u}{3e^{n+2}}\right] \\ & \leq \sum_{n \in \mathbb{Z}} C\varepsilon e^n \cdot \min\{e^{-n}, 1\}^D \frac{1}{\varepsilon^\theta} \frac{u^{-\bar{\alpha}}}{\sqrt{\log u}} \cdot e^{\alpha n + \delta|n|} \leq C\varepsilon^{1-\theta} \cdot \frac{u^{-\bar{\alpha}}}{\sqrt{\log u}} \end{aligned}$$

and Proposition (4.13) follows. \square

Lemma 4.15. *Assume that (1.6), (1.8), (1.11), (1.13) and (1.14) are satisfied. For any fixed (large) $L > 0$ there exist C such that for every (small) $\epsilon, \eta > 0$, $0 < j \leq L$ and $u \geq 2$ we have*

$$(4.16) \quad \mathbb{P}\left[\Pi_{k_u-1}B_{k_u} > \epsilon u, \Pi_{k_u-L}Y'_j \in u(1-\eta^L, 1+\eta^L)\right] \leq C\epsilon^{-\alpha}\eta^L \cdot \frac{u^{-\bar{\alpha}}}{\sqrt{\log u}}.$$

Moreover, for every $\epsilon, \eta > 0$ and $u \geq 2$

$$(4.17) \quad \mathbb{P}\left[\Pi_{k_u-1}B_{k_u} > \epsilon u, \Pi_{k_u-L}M'_L \in u(1-\eta^L, 1+\eta^L)\right] \leq CL\epsilon^{-\alpha}\eta^L \cdot \frac{u^{-\bar{\alpha}}}{\sqrt{\log u}}$$

and

$$(4.18) \quad \mathbb{P}\left\{\Pi_{k_u}B_{k_u+1} > \epsilon_0 u, \Pi_{k_u-L}Y'_L + \Pi_{k_u}B_{k_u+1} \in (1-\eta^L, 1+\eta^L)u\right\} \leq C\epsilon^{-\alpha}\eta^L \cdot \frac{u^{-\bar{\alpha}}}{\sqrt{\log u}}.$$

Proof. To prove (4.16), we use the same argument as for the proof of Proposition 4.1. Let $J_{\eta^L} = u(1-\eta^L, 1+\eta^L)$. We have

$$(4.19) \quad \begin{aligned} & \mathbb{P}[\Pi_{k_u-L}Y'_j \in J_{\eta^L}, \Pi_{k_u-1}B_{k_u} > \epsilon u] \\ &= \mathbb{P}[A_1\Pi'_{k_u-L-1}Y'_j \in J_{\eta^L}, A_1\Pi'_{k_u-2}B_{k_u} > \epsilon u] \\ &\leq \sum_{n \in \mathbb{Z}} \mathbb{P}\left[e^n \leq A_1 < e^{n+1}, A_1\Pi'_{k_u-L-1}Y'_j \in J_{\eta^L} \text{ and } \Pi'_{k_u-2}B_{k_u} > \frac{\epsilon u}{e^{n+1}}\right] \\ &\leq \sum_{n \in \mathbb{Z}} \int \mathbf{1}_{\{\frac{u}{2e^{n+1}} < s < \frac{2u}{e^n}\}} \mathbb{P}\left[e^n \leq A_1 < e^{n+1}, A_1 \in \frac{1}{s}J_{\eta^L}\right] \\ &\quad \cdot \mathbb{P}\left[\Pi'_{k_u-2}B_{k_u} > \frac{\epsilon u}{e^{n+1}}, \Pi'_{k_u-L-1}Y'_j \in ds\right] \end{aligned}$$

For s in the domain of the integral, the length of $\frac{1}{s}J_{\eta^L}$ is at most $C\eta^Le^n$. As before, $\mathbb{P}\left[e^n \leq A_1 < e^{n+1}, A_1 \in \frac{1}{s}J_{\eta^L}\right] \leq C\eta^Le^n \cdot \min\{e^{-n}, 1\}^D$. Hence the last quantity of (4.19) is dominated by

$$\sum_{n \in \mathbb{Z}} C\eta^Le^n \cdot \min\{e^{-n}, 1\}^D \cdot \mathbb{P}\left[\Pi'_{k_u-2}B_{k_u} > \frac{\epsilon u}{e^{n+1}}\right].$$

Now applying Lemma 4.3 for a fixed δ we obtain

$$\begin{aligned} \mathbb{P}[\Pi_{k_u-L}Y'_j \in J_{\eta^L}, \Pi_{k_u-1}B_{k_u} > \epsilon u] &\leq \sum_{n \in \mathbb{Z}} C\eta^Le^n \cdot \min\{e^{-n}, 1\}^D e^{\alpha n + \delta|n|} \frac{u^{-\bar{\alpha}}}{\sqrt{\log u}} \\ &\leq C\eta^L\epsilon^{-\alpha} \cdot \frac{u^{-\bar{\alpha}}}{\sqrt{\log u}} \end{aligned}$$

and (4.16) follows.

The estimate (4.17) follows immediately from (4.16) and the definition M'_L . The proof of (4.18) is similar to (4.16). \square

5. ASYMPTOTICS

Proof of Theorem 1.10. Step 1. Fix ε_0 and take u such that $\varepsilon_0 > u^{-\xi'}$. Then by Proposition 4.1

$$\begin{aligned}
 \mathbb{P}[\tau = k_u + 1] &= \mathbb{P}[M_{k_u} \leq u, M_{k_u+1} > u] \\
 &= \mathbb{P}[M_{k_u} \leq u, Y_{k_u+1} > u] \\
 &= \mathbb{P}[M_{k_u} \leq u, Y_{k_u} \in [(1 - \varepsilon_0)u, u], Y_{k_u+1} > u] \\
 &\quad + \mathbb{P}[M_{k_u} \leq u, Y_{k_u} < (1 - \varepsilon_0)u, Y_{k_u+1} > u] \\
 &= O\left(\frac{\varepsilon_0^{1-\sigma}}{\sqrt{\log u}} u^{-\bar{\alpha}}\right) + \mathbb{P}[M_{k_u} \leq u, Y_{k_u} < (1 - \varepsilon_0)u, Y_{k_u+1} > u].
 \end{aligned}
 \tag{5.1}$$

Thus is sufficient to prove that

$$\lim_{u \rightarrow \infty} u^{\bar{\alpha}} \sqrt{\log u} \mathbb{P}[M_{k_u} \leq u, Y_{k_u} < (1 - \varepsilon_0)u, Y_{k_u+1} > u] \quad \text{exists}
 \tag{5.2}$$

for some fixed arbitrary small ε_0 . Indeed, having proved (5.2) we first let $u \rightarrow \infty$ and then $\varepsilon_0 \rightarrow 0$.

Similar arguments as in the proof of Lemma 4.7 show that there are constants: large L_0 and $\eta, \eta_1 < 1$ and C possibly depending on ε_0 such that for any $L > L_0$ we have

$$\mathbb{P}[M_{k_u-L} \geq \eta^L u, \Pi_{k_u} B_{k_u+1} > \varepsilon_0 u] \leq \eta_1^L \frac{C}{\sqrt{\log u}} u^{-\bar{\alpha}}
 \tag{5.3}$$

and

$$\mathbb{P}[|Y_{k_u-L}| \geq \eta^L u, \Pi_{k_u} B_{k_u+1} > \varepsilon_0 u] \leq \eta_1^L \frac{C}{\sqrt{\log u}} u^{-\bar{\alpha}}
 \tag{5.4}$$

Therefore, choosing large (but fixed) L , to prove the main result, it is enough to show that

$$\lim_{u \rightarrow \infty} u^{\bar{\alpha}} \sqrt{\log u} \mathbb{P}[\Omega] \quad \text{exists,}
 \tag{5.5}$$

where

$$\Omega = \left\{ M_{k_u-L} < \eta^L u, |Y_{k_u-L}| < \eta^L u, M_{k_u} < u, Y_{k_u} < (1 - \varepsilon_0)u, Y_{k_u+1} > u \right\}.$$

As before, we conclude letting first $u \rightarrow \infty$ then $L \rightarrow \infty$. The limit in (5.5), if exist, has to be positive. Indeed, taking L large, we can make the upper bounds in (5.3) and (5.4) smaller than the lower bound in Proposition 3.1.

Step 2. To prove (5.5) we modify further the set Ω and as we will see, it is sufficient to replace Ω by a set Ω_2 defined below. By the definition of M_{k_u}

$$M_{k_u} = \max_{j=1, \dots, k_u} \{Y_j\} = \max \left\{ M_{k_u-L}, Y_{k_u-L} + \Pi_{k_u-L} M'_L \right\},$$

where $M'_L = \max_{j=1, \dots, L} \sum_{i=k_u-L}^j A_{k_u-L+1} \cdots A_{i-1} B_i$. Notice that M'_L has the same law as M_L .

Define the sets

$$\begin{aligned}\Omega_1 &= \left\{ M_{k_u-L} < \eta^L u, \Pi_{k_u-L} M'_L < (1 - \eta^L)u, |Y_{k_u-L}| < \eta^L u, \Pi_{k_u-L} Y'_L < (1 - \varepsilon_0 - \eta^L)u, \right. \\ &\quad \left. \Pi_{k_u-L} Y'_L + \Pi_{k_u} B_{k_u+1} > (1 + \eta^L)u \right\} \\ \Omega_2 &= \left\{ \Pi_{k_u-L} M'_L < (1 + \eta^L)u, \Pi_{k_u-L} Y'_L < (1 - \varepsilon_0 + \eta^L)u, \Pi_{k_u-L} Y'_L + \Pi_{k_u} B_{k_u+1} > (1 - \eta^L)u \right\}\end{aligned}$$

Then

$$\Omega_1 \subset \Omega \subset \Omega_2.$$

It is convenient to modify slightly Ω_1 and consider

$$\begin{aligned}\Omega'_1 &= \Omega_1 \cup \Omega''_1 \\ &= \left\{ \Pi_{k_u-L} M'_L < (1 - \eta^L)u, \Pi_{k_u-L} Y'_L < (1 - \varepsilon_0 - \eta^L)u, \right. \\ &\quad \left. \Pi_{k_u-L} Y'_L + \Pi_{k_u} B_{k_u+1} > (1 + \eta^L)u \right\}\end{aligned}$$

for

$$\begin{aligned}\Omega''_1 &= \left\{ M_{k_u-L} \geq \eta^L u, \Pi_{k_u-L} M'_L < (1 - \eta^L)u, \Pi_{k_u-L} Y'_L < (1 - \varepsilon_0 - \eta^L)u, \right. \\ &\quad \left. \Pi_{k_u-L} Y'_L + \Pi_{k_u} B_{k_u+1} > (1 + \eta^L)u \right\} \cup \left\{ |Y_{k_u-L}| \geq \eta^L u, \Pi_{k_u-L} M'_L < (1 - \eta^L)u, \right. \\ &\quad \left. \Pi_{k_u-L} Y'_L < (1 - \varepsilon_0 - \eta^L)u, \Pi_{k_u-L} Y'_L + \Pi_{k_u} B_{k_u+1} > (1 + \eta^L)u \right\}\end{aligned}$$

Notice that by (5.3) and (5.4)

$$\begin{aligned}\mathbb{P}[\Omega''_1] &\leq \mathbb{P}[M_{k_u-L} > \eta^L u, \Pi_{k_u} B_{k_u+1} > \varepsilon_0 u] + \mathbb{P}[|Y_{k_u-L}| > \eta^L u, \Pi_{k_u} B_{k_u+1} > \varepsilon_0 u] \\ &\leq \eta_1^L \frac{C_{\varepsilon_0}}{\sqrt{\log u}} u^{-\bar{\alpha}}.\end{aligned}$$

We have

$$\mathbb{P}[\Omega'_1] - \mathbb{P}[\Omega''_1] \leq \mathbb{P}[\Omega] \leq \mathbb{P}[\Omega_2]$$

We claim that $\mathbb{P}[\Omega_2 \setminus \Omega'_1] \leq C\varepsilon_0^{-\alpha} \eta^L \cdot \frac{u^{-\bar{\alpha}}}{\sqrt{\log u}}$. We have

$$\begin{aligned}\mathbb{P}[\Omega_2 \setminus \Omega'_1] &\leq \mathbb{P}\left\{ \Pi_{k_u-L} M'_L \in (1 - \eta^L, 1 + \eta^L)u, \Pi_{k_u} B_{k_u+1} > \varepsilon_0 u \right\} \\ &\quad + \mathbb{P}\left\{ \Pi_{k_u-L} Y'_L \in (1 - \varepsilon_0 - \eta^L, 1 - \varepsilon_0 + \eta^L)u, \Pi_{k_u} B_{k_u+1} > \varepsilon_0 u \right\} \\ &\quad + \mathbb{P}\left\{ \Pi_{k_u-L} Y'_L + \Pi_{k_u} B_{k_u+1} \in (1 - \eta^L, 1 + \eta^L)u, \Pi_{k_u} B_{k_u+1} > \varepsilon_0 u \right\}.\end{aligned}$$

and the claim follows from Lemma 4.15 applied to each summand. Now it suffices to prove that

$$\lim_{u \rightarrow \infty} u^{\bar{\alpha}} \sqrt{\log u} \mathbb{P}[\Omega_2] \text{ exists.}$$

Step 3. To proceed we write

$$P(u, \varepsilon, \delta, \gamma) = \mathbb{P} \left[\Pi_{k_u-L} M'_L < (1 + \varepsilon)u, \Pi_{k_u-L} Y'_L < (1 + \delta)u, \Pi_{k_u-L} Y'_L + \Pi_{k_u} B_{k_u+1} > (1 + \gamma)u \right],$$

where $\varepsilon, \delta, \gamma$ may have arbitrary signs. Notice that

$$(5.6) \quad \mathbb{P}(\Omega_2) = P(u, \eta^L, -\varepsilon_0 + \eta^L, -\eta^L).$$

We are going to prove that

$$P(u, \varepsilon, \delta, \gamma) = C_L \frac{u^{-\bar{\alpha}}}{\sqrt{\log u}} (1 + o(1))$$

for some constant C_L . In fact, it is sufficient to show that

$$(5.7) \quad \tilde{P}(u, \varepsilon, \delta, \gamma) = C_L \frac{u^{-\bar{\alpha}}}{\sqrt{\log u}} (1 + o(1)),$$

where

$$\begin{aligned} \tilde{P}(u, \varepsilon, \delta, \gamma) = \mathbb{P} \left[\Pi_{k_u-L} M'_L < (1 + \varepsilon)u, \Pi_{k_u-L} Y'_L < (1 + \delta)u, \right. \\ \left. \Pi_{k_u-L} Y'_{L+1} = \Pi_{k_u-L} Y'_L + \Pi_{k_u} B_{k_u+1} > (1 + \gamma)u, \text{ and } ue^{-(\log u)^{\frac{1}{4}}} < \Pi_{k_u-L} < ue^{(\log u)^{\frac{1}{4}}} \right]. \end{aligned}$$

Indeed

$$\begin{aligned} P(u, \varepsilon, \delta, \gamma) - \tilde{P}(u, \varepsilon, \delta, \gamma) &\leq \mathbb{P} \left[\Pi_{k_u-L} > ue^{(\log u)^{\frac{1}{4}}} \right] \\ &\quad + \mathbb{P} \left[\Pi_{k_u-L} < ue^{-(\log u)^{\frac{1}{4}}} \text{ and } \Pi_{k_u} B_{k_u+1} > (\gamma - \delta)u \right]. \end{aligned}$$

Applying the Chebychev inequality with α to the first term we have

$$\mathbb{P} \left[\Pi_{k_u-L} > ue^{(\log u)^{\frac{1}{4}}} \right] \leq e^{-\alpha(\log u)^{\frac{1}{4}}} u^{-\alpha} \lambda(\alpha)^{k_u-L} = o \left(\frac{u^{-\bar{\alpha}}}{\sqrt{\log u}} \right).$$

For the second term we choose $\beta > \alpha$ and we write

$$\begin{aligned} &\mathbb{P} \left[\Pi_{k_u-L} < ue^{-(\log u)^{\frac{1}{4}}} \text{ and } \Pi_{k_u} B_{k_u+1} > (\gamma - \delta)u \right] \\ &\leq \sum_{m \geq 0} \mathbb{P} \left[ue^{-(\log u)^{\frac{1}{4}}} e^{-(m+1)} \leq \Pi_{k_u-L} < ue^{-(\log u)^{\frac{1}{4}}} e^{-m} \right] \cdot \mathbb{P} \left[\Pi_L B > (\gamma - \delta) e^{(\log u)^{\frac{1}{4}}} e^m \right] \\ &\leq \sum_{m \geq 0} \frac{C e^{\alpha(\log u)^{\frac{1}{4}}} e^{\alpha m}}{u^{\alpha}} \lambda(\alpha)^{k_u-L} \frac{\lambda(\beta)^L \mathbb{E}|B|^{\beta}}{e^{\beta(\log u)^{\frac{1}{4}}} e^{\beta m}} \\ &= o \left(\frac{u^{-\bar{\alpha}}}{\sqrt{\log u}} \right), \end{aligned}$$

applying Chebychev with α and β respectively. The our proof is reduced to (5.7).

Step 4. Finally, notice that $\tilde{P}(u, \varepsilon, \delta, \gamma)$ is the probability of a set on which

$$ue^{-(\log u)^{\frac{1}{4}}} < \Pi_{k_u-L} < ue^{(\log u)^{\frac{1}{4}}}$$

Therefore, we may apply the Petrov theorem and we have

$$\begin{aligned}
& \tilde{P}(u, \varepsilon, \delta, \gamma) \\
&= \int \mathbb{P} \left[u \max \left(\frac{(1+\gamma)}{s_3}, e^{-(\log u)^{\frac{1}{4}}} \right) < \Pi_{k_u-L} < u \min \left(\frac{(1+\varepsilon)}{s_1}, \frac{(1+\delta)}{s_2}, e^{(\log u)^{\frac{1}{4}}} \right) \right] \\
&= \mathbb{P} \left[M'_L \in (s_1, s_1 + ds_1), Y'_L \in (s_2, s_2 + ds_2), Y'_{L+1} \in (s_3, s_3 + ds_3) \right] \\
&= \frac{C u^{-\bar{\alpha}}}{\sqrt{\log u}} (1 + o(1)) \mathbb{E} \left[\left(\left(\frac{Y'_{L+1}}{1+\gamma} \right)^\alpha - \max \left(\left(\frac{M'_L}{(1+\varepsilon)} \right)^\alpha, \left(\frac{Y'_L}{(1+\delta)} \right)^\alpha \right) \right)_+ \right]
\end{aligned}$$

The last integral is finite by moment assumption (1.8) and the conclusion follows. \square

6. PROOF OF THEOREM 1.17

First for $\beta < \alpha + \xi$ we will need the following statement which is, in fact, Lemma 3.2 with $r = D_1$.

Lemma 6.1. *Under assumptions of Theorem 1.17, there are constants D_1, C_1, C such that for $n = \frac{\log u}{\Lambda'(\beta)}$ and every $\gamma > 0$*

$$\mathbb{P}[\gamma u < \Pi_n < D_1 \gamma u, Y_n < C_1 \gamma u] \geq \frac{C \lambda(\beta)^n}{u^\beta \gamma^\beta \sqrt{\log u}}$$

Lemma 6.2. *Under assumptions of Theorem 1.17, there are constants D_2, C_2, C_3, C, m_0 be such that for any $m > m_0$ and $\varepsilon = e^{m\Lambda'(1)} < 1$*

$$\mathbb{P}[\varepsilon \leq \Pi_m \leq D_2 \varepsilon, C_3 \leq Y_m \leq C_2] \geq \frac{C \lambda(1)^m}{\varepsilon \sqrt{m}}.$$

Proof. Step 1. We change the probability space and consider the probability measure $\lambda(1)^{-1} a\mu(da, db)$. Denote by \mathbb{P}_1 the corresponding probability measure on the space of trajectories and by \mathbb{E}_1 the corresponding expected value. Then, for $S_m = \log \Pi_m$,

$$\begin{aligned}
\mathbb{P}[\varepsilon \leq \Pi_m \leq D_2 \varepsilon] &\geq \frac{\lambda(1)^m}{D_2 \varepsilon} \mathbb{E} \left[\mathbf{1}_{\{\varepsilon \leq \Pi_m \leq D_2 \varepsilon\}} \frac{\Pi_m}{\lambda(1)^m} \right] \\
&= \frac{\lambda(1)^m}{D_2 \varepsilon} \mathbb{P}_1[0 \leq S_m - m\Lambda'(1) \leq \log D_2].
\end{aligned}$$

Since S_m is a sum of iid random variables and $\mathbb{E}_1 S_m = m\Lambda'(1)$, by the local limit theorem

$$\mathbb{P}[\varepsilon \leq \Pi_m \leq D_2 \varepsilon] \geq \frac{C \lambda(1)^m}{\varepsilon \sqrt{m}}.$$

Step 2. Denote $\Pi'_{m-j-1} = A_{j+2} \dots A_m$. We have

$$\begin{aligned}
\mathbb{P}[\varepsilon \leq \Pi_m \leq D_2 \varepsilon, Y_m \geq C_2] &\leq \sum_{j=1}^{m-1} \mathbb{P} \left[\varepsilon < \Pi_m \leq D_2 \varepsilon, \Pi_j B_{j+1} > \frac{C_2}{2j^2} \right] \\
&= \sum_{j=1}^{m-1} \int \mathbb{P} \left[\varepsilon < \Pi_j a \Pi'_{m-j-1} < D_2 \varepsilon, \Pi_j b > \frac{C_2}{2j^2} \right] \mu(da, db)
\end{aligned}$$

and for every j we consider

$$P_j = \int \mathbb{P} \left[\varepsilon < \Pi_j a \Pi'_{m-j-1} < D_2 \varepsilon, \Pi_j b > \frac{C_2}{2j^2} \right] \mu(da, db).$$

Step 2a. Since $\alpha_{\min} > 1$, $\Lambda(1), \Lambda'(1) < 0$ and so we can choose $r < 1$ and β such that $r\Lambda(\beta) < \Lambda(1) - \Lambda'(1)$ and $\Lambda(\beta) < 0$. Then, by the Chebyshev inequality,

$$\begin{aligned} \sum_{j > rm} P_j &\leq \sum_{j > rm} \int \mathbb{P} \left[\Pi_j b > \frac{C_2}{2j^2} \right] \mu(da, db) \\ &\leq \sum_{j > rm} \frac{(2j^2)^\beta}{C_2^\beta} \lambda(\beta)^j \mathbb{E} B^\beta \leq C e^{rm(\Lambda(\beta) + \delta)} \\ &= o \left(\frac{\lambda(1)^m}{\varepsilon \sqrt{m}} \right) \end{aligned}$$

for a positive δ such that $r(\Lambda(\beta) + \delta) < \Lambda(1) - \Lambda'(1)$.

Step 2b. For $j \leq rm$ we write

$$\begin{aligned} \sum_{j \leq rm} P_j &\leq \sum_{j \leq rm} \sum_{k \geq 0} \int \mathbb{P} \left[\varepsilon < \Pi_j a \Pi'_{m-j-1} < D_2 \varepsilon, \frac{C_2 e^k}{2j^2 b} < \Pi_j < \frac{C_2 e^{k+1}}{2j^2 b} \right] \mu(da, db) \\ &\leq \sum_{j \leq rm} \sum_{k \geq 0} \int \mathbb{P} \left[\frac{2\varepsilon j^2 b}{C_2 a e^{k+1}} < \Pi_{m-j-1} < \frac{2D_2 \varepsilon j^2 b}{C_2 a e^k} \right] \cdot \mathbb{P} \left[\Pi_j > \frac{C_2 e^k}{2j^2 b} \right] \mu(da, db) \end{aligned}$$

To proceed further we recall the Berry-Essen theorem (see e.g. [21]) that for an i.i.d. sequence $\{X_j\}$ with variance σ^2 and finite third moment, gives

$$(6.3) \quad \sup_x \left| \mathbb{P} \left[\frac{\sum_{j=1}^m X_j - m\mathbb{E}X_1}{\sigma \sqrt{m}} < x \right] - \Phi(x) \right| \leq \frac{\overline{C} \mathbb{E}|X_1 - \mathbb{E}X_1|^3}{\sigma^3} \cdot \frac{1}{\sqrt{m}} = \frac{\gamma}{\sqrt{m}}$$

where Φ denotes the normal distribution and \overline{C} is a universal constant. Hence, changing again the probability space, we have

$$\begin{aligned}
\mathbb{P}\left[\frac{2\varepsilon j^2 b}{C_2 a e^{k+1}} < \Pi_{m-j-1} < \frac{2D_2 \varepsilon j^2 b}{C_2 a e^k}\right] \\
&\leq \frac{\lambda(1)^{m-j-1} C_2 a e^{k+1}}{2\varepsilon j^2 b} \mathbb{E}\left[\mathbf{1}_{\left\{\frac{2\varepsilon j^2 b}{C_2 a e^{k+1}} < \Pi_{m-j-1} < \frac{2D_2 \varepsilon j^2 b}{C_2 a e^k}\right\}} \frac{\Pi_{m-j-1}}{\lambda(1)^{m-j-1}}\right] \\
&= \frac{\lambda(1)^{m-j-1} C_2 a e^{k+1}}{2\varepsilon j^2 b} \mathbb{P}_1\left[m\Lambda'(1) + \log(2j^2/C_2) + \log(b/a) - (k+1) < S_{m-j-1}\right. \\
&\quad \left.< m\Lambda'(1) + \log(2D_2 j^2/C_2) + \log(b/a) - k\right] \\
&= \frac{\lambda(1)^{m-j-1} C_2 a e^{k+1}}{2\varepsilon j^2 b} \mathbb{P}_1\left[\frac{(j+1)\Lambda'(1) + \log(2j^2/C_2) + \log(b/a) - (k+1)}{\sqrt{m-j-1}}\right. \\
&\quad \left.< \frac{S_{m-j-1} - (m-j-1)\Lambda'(1)}{\sqrt{m-j-1}} < \frac{(j+1)\Lambda'(1) + \log(2D_2 j^2/C_2) + \log(b/a) - k}{\sqrt{m-j-1}}\right] \\
&\leq \frac{\gamma C_2 \lambda(1)^{m-j-1} a e^{k+1}}{2\varepsilon j^2 b \sqrt{m-j-1}}.
\end{aligned}$$

(6.3) has been used in the last inequality. For $\mathbb{P}[\Pi_j > \frac{C_2 e^k}{2j^2 b}]$ we use the Chebychev inequality with α and so

$$\begin{aligned}
\sum_{j \leq rm} P_j &\leq C \sum_{j \leq rm} \sum_{k \geq 0} \frac{C_2 \lambda(1)^{m-j-1} e^k}{\varepsilon j^2 \sqrt{m-j-1}} \frac{j^{2\alpha} \lambda(\alpha)^j}{C_2^\alpha e^{\alpha k}} \int ab^{\alpha-1} \mu(da, db) \\
&\leq C C_2^{1-\alpha} \frac{\lambda(1)^m}{\varepsilon \sqrt{m}} \sum_{j \leq rm} \left(\frac{\lambda(\alpha)}{\lambda(1)}\right)^j j^{2\alpha} \\
&\leq \frac{C}{C_2^{\alpha-1}} \frac{\lambda(1)^m}{\varepsilon \sqrt{m}}
\end{aligned}$$

Step 3. Combining first two steps and taking large C_2 we obtain

$$\mathbb{P}[\varepsilon \leq \Pi_m \leq D_2 \varepsilon, Y_m \leq C_2] \geq \frac{C \lambda(1)^m}{\varepsilon \sqrt{m}}.$$

Take parameters $a_2 > 1$, $b_1 < b_2$, $\eta > 0$ such that

$$\mathbb{P}[A \in (1, a_2), B \in (b_1, b_2)] \geq \eta > 0$$

Then, for (A_0, B_0) independent of Π_m and Y_m , we have

$$\begin{aligned}
\eta \frac{C \lambda(1)^m}{\varepsilon \sqrt{m}} &\leq \mathbb{P}[A_0 \in (1, a_2), B_0 \in (b_1, b_2)] \cdot \mathbb{P}[\varepsilon \leq \Pi_m \leq D_2 \varepsilon, Y_m \leq C_2] \\
&= \mathbb{P}[A_0 \in (1, a_2), B_0 \in (b_1, b_2) \text{ and } \varepsilon \leq \Pi_m \leq D_2 \varepsilon, Y_m \leq C_2] \\
&\leq \mathbb{P}[\varepsilon \leq \Pi_{m+1} \leq a_2 D_2 \varepsilon, b_1 \leq Y_m \leq b_2 + a_2 C_2].
\end{aligned}$$

Thus the Lemma follows for $C_3 = b_1$ and for D_2, C_2 replaced by $a_2 D_2, b_2 + a_2 C_2$. \square

Proof of Theorem 1.17. Let $\beta > \alpha$ be very close to α and define p by the relation

$$(6.4) \quad \Lambda'(\beta) = \frac{\Lambda'(\alpha)}{p}.$$

Since $\Lambda''(\beta) > 0$, we have $p < 1$. The precise value of β will be chosen later on. Take $n = pk_u = \frac{\log u}{\Lambda'(\beta)}$. Let $q = 1 - p$, $m = qk_u = k_u - n$. We write

$$Y_{k_u} = Y_n + \Pi_{n-1}A_nY'_m = Y_{n-1} + \Pi_{n-1}B_n + \Pi_{n-1}A_nY'_m.$$

for $Y'_m = \sum_{i=n+1}^{k_u} A_{n+1} \dots A_{i-1} B_i$. Then Y'_m has the same law as Y_m . We denote also $\Pi'_j = A_{n+1} \dots A_{n+j}$, $B'_m = B_{k_u}$ and we consider the set

$$\Omega = \left\{ \gamma u < \Pi_{n-1} < D_1 \gamma u, Y_{n-1} \leq C_1 \gamma u, C_3 \leq Y'_m \leq C_2, \varepsilon < \Pi'_{m-1} < D_2 \varepsilon, b_1 < B'_m < b_2 \right\},$$

where γ, D_1 , are parameters given in Lemma 6.1, C_2, C_3, D_2 are described in Lemma 6.2 $\varepsilon = e^{m\Lambda'(1)}$, b_1 and b_2 are chosen as in (1.20).

By Lemmas 6.1 and 6.2

$$(6.5) \quad \mathbb{P}(\Omega) \geq \frac{C\lambda(\beta)^n}{u^\beta} \frac{1}{\sqrt{n}} \frac{\lambda(1)^m}{\varepsilon\sqrt{m}}.$$

We write

$$(6.6) \quad \begin{aligned} & \mathbb{P}[Y_{k_u-1} \leq u, Y_{k_u} > u] \\ &= \mathbb{P}\left[Y_{n-1} + \Pi_{n-1}B_n + \Pi_{n-1}A_nY'_m - \Pi_{n-1}A_n\Pi'_{m-1}B_m \leq u, \right. \\ & \quad \left. Y_{n-1} + \Pi_{n-1}B_n + \Pi_{n-1}A_nY'_m > u\right] \\ &\geq \mathbb{P}\left[\left\{ \frac{u - Y_{n-1} - \Pi_{n-1}B_n}{\Pi_{n-1}Y'_m} < A_n \leq \frac{u - Y_{n-1} - \Pi_{n-1}B_n}{\Pi_{n-1}(Y'_m - \Pi'_{m-1}B'_m)}, b_1 \leq B_n \leq b_2 \right\} \cap \Omega\right] \end{aligned}$$

Notice that on the set Ω , for $b_1 \leq B_n \leq b_2$ we have

$$\begin{aligned} \frac{u - Y_{n-1} - \Pi_{n-1}B_n}{\Pi_{n-1}(Y'_m - \Pi'_{m-1}B'_m)} &< \frac{u}{\gamma u C_3/2} = \frac{2}{\gamma C_3} =: C_+ \\ \frac{u - Y_{n-1} - \Pi_{n-1}B_n}{\Pi_{n-1}Y'_m} &> \frac{u - C_1 \gamma u - \gamma u b_1}{D_1 \gamma u C_2} = \frac{1 - C_1 \gamma - \gamma b_1}{D_1 \gamma C_2} =: C_-. \end{aligned}$$

Here we have used $B'_m \Pi'_{m-1} \leq D_2 b_2 \varepsilon \leq \frac{C_3}{2}$ for large m . Moreover on Ω

$$\begin{aligned} & \frac{u - Y_{n-1} - \Pi_{n-1}B_n}{\Pi_{n-1}(Y'_m - \Pi'_{m-1}B'_m)} - \frac{u - Y_{n-1} - \Pi_{n-1}B_n}{\Pi_{n-1}Y'_m} \\ & \geq \frac{u - Y_{n-1} - \Pi_{n-1}B_n}{\Pi_{n-1}} \cdot \frac{\Pi'_{m-1}B'_m}{Y'_m(Y'_m - \Pi'_{m-1}B'_m)} \\ & \geq \frac{1 - C_1 \gamma - \gamma b_1}{D_1 \gamma} \cdot \frac{\varepsilon b_1}{C_2^2} = d_1 \varepsilon. \end{aligned}$$

and we have $C_- > 0, d_1 > 0$ if we take $(C_1 + b_1)\gamma \leq \frac{1}{2}$.

The set Ω is independent on A_n, B_n . By (6.6) for fixed value of $b = B_n$, we can take any A_n in (6.6) from some interval $I_{d_1\varepsilon, b} \subset (C_-, C_+)$ (depending on b) of length at least $d_1\varepsilon$. In view of 1.20 $d_2 = \inf_{C_- < a < C_+} f(a)$ is strictly positive. Then, by (6.5)

$$\mathbb{P}[Y_{k_u-1} < u, Y_{k_u} > u] \geq \inf_{I_{d_1\varepsilon, b} \subset (C_-, C_+)} \mathbb{P}[A \in I_{d_1\varepsilon}] \mathbb{P}[\Omega] \geq C d_1 d_2 \cdot \frac{\lambda(\beta)^n}{u^\beta} \frac{1}{\sqrt{n}} \frac{\lambda(1)^m}{\sqrt{m}}$$

Since \sqrt{nm} is of order $\log u$, to finish the proof we have to justify that

$$\frac{\lambda(\beta)^n \lambda(1)^m}{u^\beta} \geq \frac{\lambda(\alpha)^{k_u}}{u^\alpha} \cdot u^\delta$$

for some δ . In other words we want to show

$$\left(\frac{\lambda(\beta)}{\lambda(\alpha)} \right)^n \left(\frac{\lambda(1)}{\lambda(\alpha)} \right)^m \geq u^{\beta-\alpha} u^\delta.$$

Choose p in (6.4) such that $\beta - \alpha < \eta$ for $\eta = \frac{\log \mu}{\Lambda'(\alpha)}$ and $\mu = \frac{\lambda(1)}{\lambda(\alpha)} > 1$. Then, by the Taylor expansion of Λ , since $\Lambda''(\beta) > 0$, $\Lambda(\beta) - \Lambda(\alpha) \geq \Lambda'(\alpha)(\beta - \alpha)$ and so

$$\begin{aligned} \left(\frac{\lambda(\beta)}{\lambda(\alpha)} \right)^n \left(\frac{\lambda(1)}{\lambda(\alpha)} \right)^m &= e^{n(\Lambda(\beta) - \Lambda(\alpha))} \mu^m \\ &\geq e^{n(\beta - \alpha)\Lambda'(\alpha)} \mu^m \end{aligned}$$

But $n(\beta - \alpha)\Lambda'(\alpha) = (\beta - \alpha)p \log u$ and $\mu^m = e^{\log \mu \cdot \frac{q \log u}{\Lambda'(\alpha)}} = e^{q\eta \log \mu}$. Hence

$$\begin{aligned} \left(\frac{\lambda(\beta)}{\lambda(\alpha)} \right)^n \left(\frac{\lambda(1)}{\lambda(\alpha)} \right)^m &\geq u^{p(\beta - \alpha)} u^{\eta q} \\ &= u^{\beta - \alpha} u^{(\eta - (\beta - \alpha))q}, \end{aligned}$$

which completes proof of the Theorem. \square

Remark 6.7. The assumption on $f_A(a)$ in (1.20) can be weakened to $f_A(a) \geq c > 0$ on some interval (a_1, a_2) with $a_1 > 1$. The proof is similar. It requires only a more careful definition of Ω . If $f_A(a)da$ contains nontrivial absolutely continuous part, by Steinhaus theorem, its convolution power has to satisfy the condition above. We leave the details for the reader.

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